Homework 10

Complex analysis, lecture 4

Due December 1, 2025 by 11:59 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software/generative AI and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

Problem 1. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, and let $h : \partial\mathbb{D} \to \mathbb{C}$ be a continuous function, with Poisson integral

$$\tilde{h}(z) = \tilde{h}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\phi})P_r(\theta - \phi)\, d\phi,$$

where $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$ is the Poisson kernel. Show that

$$\frac{\partial\tilde{h}}{\partial z} = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\phi}) \cdot \frac{e^{i\phi}}{(e^{\phi i} - z)^2}\, d\phi$$

and

$$\frac{\partial\tilde{h}}{\partial\overline{z}} = \frac{1!}{2\pi}\int_0^{2\pi} h(e^{i\phi}) \cdot \frac{e^{-i\phi}}{(e^{-\phi i} - \overline{z})^2}\, d\phi.$$

Hint: derive and use the identity

$$P_r(\theta - \phi) = 1 + \frac{ze^{-i\phi}}{1 - ze^{-i\phi}} + \frac{\overline{z}e^{i\phi}}{1 - \overline{z}e^{i\phi}},$$

where $z = re^{i\theta}$.

The above problem is directed towards Objective 13 (the Poisson integral formula and harmonic functions).

Problem 2. Let $\Omega \subset \mathbb{C}$ be a domain $g(t,z)$ be a continuous function for $a \le t \le b$ and $z \in \Omega$. For each fixed t, suppose that $g_t(z) = g(t,z)$ is a harmonic function of z. Show that

$$G(z) = \int_a^b g(t,z)\,dt$$

is a harmonic function on Ω.

Note that although it is tempting to directly differentiate under the integral sign, it is not immediately obvious that the partial derivatives of g_t vary continuously with t, which is required in order to differentiate under the integral sign. Instead, use the characterization of harmonic functions by the mean value property.

The above problem is directed towards Objective 13 (the Poisson integral formula and harmonic functions).

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.