Lecture 7: arc length

Calculus II, section 3
February 21, 2022

Now that we know a variety of methods for computing integrals, our next unit is on *applications* of integrals. In other words the main question for the next few weeks is not so much how to integrate but what to integrate. Our goal is to be able to take a problem, understand what is going on and what integration might be necessary, and compute the resulting integral, so applying fixed formulas typically will not be good enough. (That said, we'll encounter some formulas too.)

Today, we're going to look at a particular application: finding lengths of curves (or segments of them). For a straight line, we can do this by Pythagoras: for example, if our curve is given by $y = px + q$, we can find the length of the segment between $x = a$ and $x = b$ by computing the coordinates of the endpoints, $(a, pa + q)$ and $(b, pb + q)$, and applying the Pythagorean theorem to get $\sqrt{(b-a)^2 + (pb + q - pa - q)^2} = \sqrt{(b-a)^2 + p^2(b-a)^2} = (b-a)\sqrt{1 + p^2}$. What about for curves?

Suppose we have a curve defined by the equation $y = f(x)$, for a smooth function f. If we zoom in close enough on a point x, this curve is approximated by a straight line near x—this is the foundation of all of calculus! Therefore if we move by a small amount h, using the linear approximation $f(x+h) \approx f(x) + f'(x)h$ we get that the distance between $(x, f(x))$ and $(x+h, f(x+h))$ is approximately $\sqrt{(x+h-x)^2 + (f(x+h) - f(x))^2} = \sqrt{h^2 + h^2 f'(x)^2} = h\sqrt{1 + f'(x)^2}$. We can then break our curve into intervals of length h, do this process for each of them, and add them all up; upon taking the limit as $h \to 0$, this gives exactly

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Let's take an example. What is the length of the curve $y = x^{3/2}$ between $x = 0$ and $x = 4$?

We can do exactly this process. Using our general formula, what we get is

$$\int_0^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx.$$

This is the kind of integral we know how to do: it should come out to $\frac{4}{9} \cdot \frac{2}{3}(1 + \frac{9}{4}x)^{3/2}\Big|_0^4 = \frac{8}{27}(10^{3/2} - 1) \approx 9.07$. The coordinates of the endpoints are $(0,0)$ and $(4,8)$, and so the straight-line distance between them is $\sqrt{4^2 + 8^2} = \sqrt{16 + 64} = \sqrt{80} \approx 8.94$, so this is actually quite close to being a straight line.

This square root in the integral often makes things difficult, and in practice many times the integral for arc length has no closed form in terms of simpler functions and can only be computed numerically. Since we like to be able to compute things by hand, we'll focus on examples that turn out to be workable, but you should be aware that (like for integrals generally) in general the situation is much worse.

Another example we can check here is finding the length of that straight line by our formula, instead of directly by Pythagoras. This had better come out the same. Indeed, here our line is just $y = 2x$, and so the formula gives

$$\int_0^4 \sqrt{1+2^2}\,dx = 4\sqrt{5} = \sqrt{16\cdot 5} = \sqrt{80}$$

as above. We can do this with a more general line: if $y = px + q$ and we want to find the length between $x = a$ and $x = b$, then our formula gives

$$\int_a^b \sqrt{1+p^2}\,dx = (b-a)\sqrt{1+p^2}$$

just as applying Pythagoras's formula directly did above.

We've previously applied integration, together with trigonometric substitution, to find the area of a circle. Let's use this new method to find the perimeter of a circle as well. Just as for area, it's easiest to instead find a half or a quarter of the perimeter and then use the symmetries of a circle to finish; let's do a quarter. The integral we want turns out (if you're reading these notes: why?) to be

$$\begin{aligned}\int_0^r \sqrt{1+\left(\frac{-2x}{2\sqrt{r^2-x^2}}\right)^2}\,dx &= \int_0^1 \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\ &= \int_0^1 \frac{r}{\sqrt{1-x^2}}\,dx \\ &= r\sin^{-1}(1) - r\sin^{-1}(0) \\ &= \frac{\pi}{2}r.\end{aligned}$$

Therefore the perimeter of a circle is $2\pi r$ as desired. (We could also get this by differentiating the area of a circle, like on homework 2.)

You can apply the same method to compute the perimeter of an ellipse, e.g. $y = 2\sqrt{1-x^2}$. However the nice simplification that happens in the case of a circle does not happen here, and the integral has no expression in terms of simpler functions, though it can be computed numerically. Instead, it is what is called an *elliptic integral*, which have connections to other areas of math—for example, their inverse functions are called elliptic functions, and are used in complex analysis and the theories of elliptic curves and modular forms.

We'll talk more about this next week, but we can also look at parametric curves, where we specify both x and y in terms of some third variable t. This contains curves of the form $y = f(x)$, by just setting $x = t$ and $y = f(t)$, but also more general curves: for example, we can take the unit circle $x = \cos t$, $y = \sin t$, which cannot be written with y as a single function of x. A parametrization also contains more information than a function, i.e. "when" we are at a given point (this can be either a good thing or a bad thing, since sometimes we only care about the curve and this extra information is superfluous).

If we are handed a parametric curve $x = x(t)$, $y = y(t)$, we can also compute its arc length between $t = t_0$ and $t = t_1$, and doing so may elucidate our process above. If t changes by a small amount h, then x changes by roughly $x'(t)h$ and y by $y'(t)h$, using the linear approximations; therefore the arc length changes by approximately $h\sqrt{x'(t)^2 + y'(t)^2}$, or $h\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Compare to the way we did it earlier: there we could write this infinitesimal change in the same way as $h\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, which is exactly what we would get from this method by setting $x = t$. If we write $h = dt$, the infinitesimal change in t, then this is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, and so we get the total length by integrating to get

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

and our previous formula is just a special case of this.

Now for the circle $x = r\cos t$, $y = r\sin t$ we can compute the perimeter of the whole circle at once, not just a quarter or half, by taking t from (say) 0 to 2π. We have $x'(t) = -r\sin t$ and $y'(t) = r\cos t$, so the arc length is

$$\int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt = \int_0^{2\pi} r\,dt = 2\pi r$$

much more easily. (We still can't solve the elliptic integral, but this approach does make it easier to approximate numerically for those who like such things.)

We could also define the infinitesimal speed to be the rate at which the point (x, y) is moving with respect to t. For those of you who have seen some multivariable calculus, this is the norm of the gradient $|\nabla(x, y)|$. More concretely, it is the derivative of the arc length, which by definition is just $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$!

You can imagine how this formula might extend to higher dimensions. It also turns out to be the basis of a lot of geometry, e.g. Riemannian geometry, where we look at different ways of defining the distance. This comes up very frequently in relativity, with weirder metrics: for example, if we take time to be a coordinate as well, the "simplest" relativistic metric (the Minkowski metric, for totally flat spacetime) is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, i.e. speed is given by $\sqrt{-1 + x'(t)^2 + y'(t)^2 + z'(t)^2}$, and changing this formula gives rise to different and weirder spacetimes, out of which you can derive the equations for gravity and motion using Einstein's equations.