Lecture 6: improper integrals

Calculus II, section 3
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So far, when integrating an important thing to check is that the function we're integrating is defined everywhere on the interval in question—for example, we couldn't integrate $\frac{1}{x}$ from 0 to 1 because it isn't defined at 0. Indeed, if we tried to do so anyway and formally integrated, we'd find that this is the antiderivative of $\frac{1}{x}$, namely $\log x$, evaluated at 1 and at 0, so $\log 1 - \log 0 = +\infty$. This makes sense from the graph of $y = \frac{1}{x}$, since it looks like it has infinite area between 0 and 1.

Sometimes, though, weird things will happen. Consider $f(x) = \frac{1}{\sqrt{x}}$. Once again, this is defined and continuous for $x > 0$ but not at $x = 0$, so $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ doesn't make sense. However, we could still try to formally integrate it, and what we get is that by the power rule the antiderivative of $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ is $2\sqrt{x}$, so this integral would be

$$2\sqrt{1} - 2\sqrt{0} = 2.$$

This suggests that even though $\frac{1}{\sqrt{x}}$ goes to infinity at 0, the area under its graph between 0 and 1 is still finite (and is 2).

In and of itself, this actually isn't a problem: it is perfectly possible to have a region with infinite bounds but finite area. However, as we usually understand it this integral still doesn't make sense, because it involves evaluating $\frac{1}{\sqrt{x}}$ at $x = 0$. To solve this, we replace the lower bound with some positive real number a, and then take the limit as $a \to 0$: we *define* the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ to be $\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx$. By the fundamental theorem of calculus, this is just $\lim_{a\to 0^+} 2(\sqrt{1} - \sqrt{a}) = 2 - 2\lim_{a\to 0^+} \sqrt{a} = 2$, the same as plugging in $a = 0$, because the antiderivative $2\sqrt{x}$ of $\frac{1}{\sqrt{x}}$ is right-continuous at $x = 0$.

More generally, whenever we have a function $f(x)$ and an interval $[a, b]$ where $f(x)$ is defined everywhere except at one endpoint, say a, we can define the integral to be

$$\int_a^b f(x)\,dx = \lim_{a' \to a^+} \int_{a'}^b f(x)\,dx.$$

Of course, if b is the point at which the function is not defined we can similarly define

$$\int_a^b f(x)\,dx = \lim_{b' \to b^-} \int_a^{b'} f(x)\,dx.$$

If f is undefined at both points, we can define the integral by a double limit:

$$\int_a^b f(x)\,dx = \lim_{a' \to a^+} \lim_{b' \to b^-} \int_{a'}^{b'} f(x)\,dx.$$

These are called improper integrals.

As we saw with $f(x) = \frac{1}{x}$, there is no guarantee that these improper integrals will be well-defined or finite either: in that case the antiderivative is $\log x$ and so we get $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\log 1 - \log a = -\lim_{a\to 0^+}\log a$, which goes to $+\infty$. However, it does enlarge the class of integrals we can define.

We've seen that $\int_0^1 \frac{1}{x}\,dx$ does not exist even in this larger sense, but $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ does, and is equal to a finite number 2. We might then ask: for what positive real numbers p does $\int_0^1 \frac{1}{x^p}\,dx$ exist?

Well, antidifferentiating gives $\frac{x^{1-p}}{1-p}$, so we have

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\frac{1}{1-p} - \frac{a^{1-p}}{1-p}.$$

If $1-p > 0$, i.e. $p < 1$, then a^{1-p} tends to 0 as $a \to 0$, and so this integral converges to $\frac{1}{1-p}$, recovering 2 in the special case where $p = \frac{1}{2}$. If $1-p < 0$, though (i.e. $p > 1$), then $a^{1-p} \to \infty$ as $a \to 0$, so this is not well-defined; and we've seen that in the intermediate case $p = 1$ it is also not well-defined.

Instead of having infinite vertical bounds, we could also consider improper integrals over infinite horizontal bounds, i.e. infinite intervals. For example, consider

$$\int_1^\infty \frac{1}{x^2}\,dx,$$

which makes sense as an area but not formally since we can't look at $\frac{1}{x^2}$ "at infinity." Instead, we again take the limit: we define

$$\int_1^\infty \frac{1}{x^2} = \lim_{N\to\infty}\int_1^N \frac{1}{x^2}\,dx = \lim_{N\to\infty} -\frac{1}{N} + \frac{1}{1} = 1.$$

This is the same formalism we needed to rigorously define our extended definition of the factorial:

$$n! = \int_0^\infty x^n e^{-x}\,dx = \lim_{N\to\infty}\int_0^N x^n e^{-x}\,dx.$$

Let's again try this for $\frac{1}{x^p}$: we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{N\to\infty}\int_1^N \frac{1}{x^p}\,dx = \lim_{N\to\infty}\frac{N^{1-p}}{1-p} - \frac{1}{1-p},$$

which is finite and equal to $-\frac{1}{1-p} = \frac{1}{p-1}$ if $1-p < 0$, i.e. $p > 1$, and goes to infinity if $1-p > 0$, i.e. $p < 1$. (For the reader: what about if $p = 1$?)

In this case, we take the antiderivative, which is straightforward, and then compute its limit. However, this is not always how improper integrals work in practice. For example, with the integral for $n!$, integrating by parts gives

$$\begin{aligned} n! &= \lim_{N\to\infty} \int_0^N x^n e^{-x}\,dx \\ &= \lim_{N\to\infty} \left(-x^n e^{-x}\Big|_0^N + n\int_0^N x^{n-1}e^{-x}\,dx \right) \\ &= \lim_{N\to\infty} \left(-N^n e^{-N} + n\int_0^N x^{n-1}e^{-x}\,dx \right). \end{aligned}$$

If we repeat this integration by parts, we get

$$\lim_{N\to\infty} \left(-N^n e^{-N} - nN^{n-1}e^{-n} - n(n-1)N^{n-2}e^{-n} - \cdots \right)$$

which even ignoring any integral term we end up with is hard to evaluate. However, if we take the limit first, then we find that all these terms vanish: $\lim_{N\to\infty} N^n e^{-N} = 0$, and so we only have the integral terms and so get our nice recurrence. This is an instance of a not-uncommon phenomenon: even when we can't compute the indefinite integral, we can sometimes still compute a definite one over an infinite range. Another example is the integral $\int_0^\infty e^{-x^2}\,dx$ which we needed in order to find the value of $(\frac{1}{2})!$.

When computing improper integrals, we have to be very careful: although it's (potentially) okay for the integrand to be undefined at the endpoints, so long as the corresponding limit exists, it has to be defined everywhere else on the interval. For example, consider

$$\int_0^\infty \frac{1}{x(\log x)^2}\,dx.$$

How can we integrate this?

Let's u-substitute: if $u = \log x$, then $du = \frac{dx}{x}$ and so this integral is (discounting the bounds as usual) $\int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\log x} + C$. Therefore the improper integral is

$$\int_0^\infty \frac{1}{x(\log x)^2}\,dx = \lim_{N\to\infty}\lim_{a\to 0} \int_a^N \frac{1}{x(\log x)^2}\,dx = \lim_{N\to\infty}\lim_{a\to 0} -\frac{1}{\log N} + \frac{1}{\log a}.$$

We have $\lim_{N\to\infty} -\frac{1}{\log N} = 0$ since $\log N$ tends to infinity as $N \to \infty$, and $\lim_{a\to\infty} \frac{1}{\log a} = 0$ since $\log a$ tends to negative infinity as $a \to 0$. Therefore this whole integral must just be zero.

But this cannot be possible: the function we are integrating, $\frac{1}{x(\log x)^2}$, is never zero or negative on $(0, \infty)$, so how can the total area under the curve be zero? It can only be because this integral is incorrect: it has an extra place where it is not defined that we have not accounted for. (Where is that place?)

We calculated $\int_0^1 \frac{1}{x^p}\,dx$ and $\int_1^\infty \frac{1}{x^p}\,dx$, so what about combining them? Can we talk about $\int_0^\infty \frac{1}{x^p}\,dx$?

We can't: if $p \leq 1$, then the term on $(1, \infty)$ is undefined, and if $p \geq 1$ then the term on $(0, 1)$ is undefined. Thus for these kinds of functions we can only look at one direction going to infinity at a time.

There are other functions for which we can look at more, though. Let's amend the above function $\frac{1}{x(\log x)^2}$ to make it work better. We still want it to go to infinity at $x = 0$, so we keep the factor of x; but we don't want to have to worry about $\log x = 0$, so we replace $(\log x)^2$ by $(\log x)^2 + 1$. In other words, we want to compute the integral $\int_0^\infty \frac{1}{x((\log x)^2+1)}\, dx$. How should we do this?

Well, u-substituting worked well before, so let's try it again: if $u = \log x$, so $du = \frac{dx}{x}$, then this integral is $\int \frac{1}{u^2+1}\, du$, which we know is $\tan^{-1}(u) + C = \tan^{-1}(\log x) + c$. Therefore the improper integral is

$$\lim_{N\to\infty} \lim_{a\to 0} \tan^{-1}(\log N) - \tan^{-1}(\log a).$$

As $N \to \infty$, $\log N \to \infty$ and so $\tan^{-1}(\log N)$ approaches $\frac{\pi}{2}$; on the other hand as $a \to 0$, $\log a \to -\infty$ and so $\tan^{-1}(\log a)$ tends to $-\frac{\pi}{2}$. Therefore the total integral exists and is simply π.

In general, we might want to know whether an improper integral converges, i.e. exists and is equal to a finite number, or diverges. There are three ways an improper integral can diverge:

1. the integrand fails to exist at some point in the interior of the interval, as for $\int_0^\infty \frac{1}{x(\log x)^2}\, dx$

2. the integrand goes to infinity "too quickly" at one of the bounds, as for $\int_0^1 \frac{1}{x^2}$;

3. the integrand goes to zero "to slowly" (or not at all) as $x \to \infty$ in the case of an infinite interval, as for $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$.

For example, $f(x) = \frac{1}{x^p}$ exists on all of $(0, \infty)$; for large p (in particular, $p > 1$) we should expect that $f(x)$ goes to zero "quickly" as $x \to \infty$, and goes to infinity "quickly" as $x \to 0$. Conversely for small p (in particular $p < 1$) we expect that $f(x)$ goes to zero, resp. infinity, "slowly" as $x \to \infty$, resp. as $x \to 0$.

This is a useful intuition, but it's not very technical and doesn't prove anything. For a more concrete test, we use the integrals we can calculate explicitly, such as $\frac{1}{x^p}$, and the comparison test, which is the following principle: if we have two functions $f(x)$ and $g(x)$ on some interval (a, b), and $0 \leq f(x) \leq g(x)$ for $x \in (a, b)$, then the area under f is bounded by the area under g. In particular, if we know that $\int_a^b g(x)\, dx$ converges, then so does $\int_a^b f(x)\, dx$; conversely, if we know that $\int_a^b f(x)\, dx$ diverges, then so does $\int_a^b g(x)\, dx$. Here a and b can be real numbers or $\pm\infty$.

For example: does the integral $\int_1^\infty \frac{1}{x^2+4x}\, dx$ converge or diverge? This is actually an integral we could compute explicitly, but there is an easier way: for every $x > 0$, we have $x^2 + 4x > x^2$ and therefore $\frac{1}{x^2+4x} \leq \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\, dx$ converges, by comparison we can immediately conclude that this integral converges too. (We have no idea what it is other than that it's positive and less than $\int_1^\infty \frac{1}{x^2}\, dx = 1$, but it definitely exists.)

Similarly, we could compute $\int_1^\infty \frac{1}{\sqrt{x}+1}\,dx$ by hand, or we could apply the comparison test. The obvious comparison is to $\frac{1}{\sqrt{x}}$, but it's the wrong way: we have $\frac{1}{\sqrt{x}+1} \leq \frac{1}{\sqrt{x}}$, because $\sqrt{x} < \sqrt{x}+1$, and this only tells us that this integral is bounded by $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$, which diverges anyway so we've learned nothing. However, on the interval $[1,\infty)$ it works out: since $\sqrt{x} \geq 1$, we have $\sqrt{x}+1 \leq \sqrt{x}+\sqrt{x} = 2\sqrt{x}$, and so $\frac{1}{\sqrt{x}+1} \geq \frac{1}{2\sqrt{x}}$, and since the integral of the latter diverges it follows that the former integral diverges as well.

We could verify in this way as well that $\int_0^\infty \frac{1}{x((\log x)^2+1)}\,dx$ converges, by splitting it up. On the interval $[2,\infty)$, we have $\frac{1}{x((\log x)^2+1)} \leq \frac{1}{x(\log x)^2}$, which is straightforward to antidifferentiate as above to $-\frac{1}{\log x}$; since we've chosen a lower bound of 2, this antiderivative is well-defined all the way down to 2 and goes to zero as $x \to \infty$, so this part converges. On the interval $(\frac{1}{2}, 2)$, the integral is actually proper since everything is well-defined; and on $(0, \frac{1}{2})$ we have $\frac{1}{x((\log x)^2+1)} \leq \frac{1}{x(\log x)^2}$ again and so can apply the same bound. From this process we see that we're actually just using the same good properties of $\frac{1}{x(\log x)^2}$ and just fixing the defect at $x = 1$ that makes it diverge.