Lecture 5: partial fractions

Calculus II, section 3
February 2, 2022

Using just linearity and the power rule, we can integrate any polynomial; we can extend this to sums of terms x^n for any n, positive or negative. The antiderivative of a polynomial is always a polynomial, but once we allow negative powers we may have logarithmic terms. Today we want to generalize this: can we integrate *rational* functions, i.e. ratios of polynomials?

Sometimes this works out very nicely. For example,

$$\int \frac{2x}{x^2-4}\,dx = \int \frac{du}{u} = \log u = \log(x^2-4)$$

where $u = x^2 - 4$. Sometimes it is more complicated: how could we integrate $\frac{x+3}{x^2-4}$?

One thing to notice is that the denominator factors: $x^2 - 4 = (x+2)(x-2)$. Therefore we could imagine that this fraction is the sum of two fractions $\frac{A}{x+2}$ and $\frac{B}{x-2}$ for some terms A and B. If so, this would be good: we've at least simplified our rational function, and if A and B are constants this is now something we know how to integrate.

Suppose that we have such A and B, i.e.

$$\frac{x+3}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}.$$

Multiplying everything by $x^2 - 4 = (x+2)(x-2)$, this is

$$x + 3 = A(x-2) + B(x+2).$$

We need this to be true for *every* x. We could try and solve more formally, but one convenient way to solve this equation is to notice that the right-hand side simplifies when $x = \pm 2$: if $x = 2$, we have

$$5 = A \cdot 0 + B \cdot 4,$$

i.e. $4B = 5$ and so $B = \frac{5}{4}$, and if $x = -2$ then we have

$$1 = A \cdot (-4) + B \cdot 0,$$

i.e. $A = -\frac{1}{4}$. Therefore

$$\frac{x+3}{x^2-4} = \frac{1}{4}\left(-\frac{1}{x+2} + \frac{5}{x-2}\right)$$

and so integrating

$$\int \frac{x+3}{x^2-4}\,dx = -\frac{1}{4}\int \frac{1}{x+2}\,dx + \frac{5}{4}\int \frac{1}{x-2}\,dx = \frac{5\log(x-2) - \log(x+2)}{4} + C.$$

(Perhaps we should say something about absolute values here, but we won't.)

This is the method of partial fractions: given a ratio of polynomials, we decompose it into a sum of simpler rational functions of the form $\frac{1}{\text{something}}$, which are hopefully easy to integrate.

Note that in the example above, it was critical that we could factor the denominator in order to get the correct decomposition.

A different sort of requirement is that the numerator of the rational function we want to integrate must have lower degree than the denominator. This is to make sure that when we do our decomposition, we end up with constant numerators for each term; if the numerators aren't constants, the integrals are harder (though maybe still doable!).

This requirement is pretty easy to get around, though. Consider

$$\int \frac{2x^2+3x+1}{x^2+x-2}\,dx.$$

We can factor the denominator as $(x+2)(x-1)$, but the numerator and denominator have the same degree so the method as above won't work nicely. To solve this, observe that the coefficient of the top-degree term in the numerator is 2, while in the denominator it is 1. Since they have the same degree, this means we can write the numerator as twice the denominator plus some lower degree terms:

$$2x^2+3x+1 = 2(x^2+x-2)+x+5,$$

so

$$\frac{2x^2+3x+1}{x^2+x-2} = \frac{2(x^2+x-2)+x+5}{x^2+x-2} = 2+\frac{x+5}{x^2+x-2}.$$

Integrating 2 is easy, so what's left is something amenable to the method above. Let's do it for the extra practice: $x^2+x-2=(x+2)(x-1)$, so we're looking for A and B such that

$$\frac{x+5}{x^2+x-2} = \frac{A}{x+2}+\frac{B}{x-1},$$

or equivalently

$$x+5 = A(x-1)+B(x+2).$$

Setting $x=1$ gives $6=3B$ and so $B=2$, and setting $x=-2$ gives $3=-3A$ and so $A=-1$, so

$$\begin{aligned}\int \frac{2x^2+3x+1}{x^2+x-2}\,dx &= \int 2+\frac{x+5}{x^2+x-2}\,dx\\ &= \int 2-\frac{1}{x+2}+\cdot\frac{2}{x-1}\,dx\\ &= 2x-\log(x+2)+2\log(x-1)+C.\end{aligned}$$

The requirement that we be able to factor the denominator is a little trickier. Suppose we have something like

$$\int \frac{2x+3}{x^2+1}\,dx.$$

We could factor the denominator using complex numbers: $x^2+1=(x+i)(x-i)$, where $i=\sqrt{-1}$. This introduces many complications, though: for one thing, then we have to worry about the logarithm of complex numbers, and although this approach is possible it's more in-depth than we'd like to do.

Fortunately, there's an easier approach: split it up! We can decompose our integral in the most obvious way:

$$\int \frac{2x+3}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx.$$

This first integral we can solve by substitution: if $u=x^2+1$, then $du=2x\,dx$ and so this is just $\int \frac{du}{u}=\log u=\log(x^2+1)$. The second integral we know how to do by trigonometric substitution: if $x=\tan\theta$, then $dx=\sec^2\theta\,d\theta=(x^2+1)\,d\theta$ and so this integral is just $3\theta=3\tan^{-1}(x)$, so in all our integral is just $\log(x^2+1)+3\tan^{-1}(x)$ (up to a constant as always).

What if we have a different non-factorable polynomial in the denominator? For example, how about $\frac{x}{x^2-2x+5}$?

We cannot factor x^2-2x+5, but what we can do is complete the square: we have $x^2-2x+5=(x-1)^2+4$, and so if $u=x-1$ then we have

$$\int \frac{x}{x^2-2x+5}\,dx = \int \frac{u+1}{u^2+4}\,du = \int \frac{u}{u^2+4}\,du + \int \frac{1}{u^2+4}\,du.$$

We can find this first integral by another substitution: set $v=u^2+4$, so that $dv=2u\,du$. Then this first integral is $\frac{1}{2}\int \frac{dv}{v}=\frac{1}{2}\log v=\frac{1}{2}\log(u^2+4)=\frac{1}{2}\log((x-1)^2+4)=\frac{1}{2}\log(x^2-2x+5)$. For the second integral, we do a trigonometric substitution as above: set $u=2\tan\theta$, so that the integral becomes

$$\int \frac{1}{4(\tan^2\theta+1)}2\sec^2\theta\,d\theta = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}(u) = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right),$$

and in all we have

$$\int \frac{x}{x^2-2x+5}\,dx = \frac{1}{2}\left(\log(x^2-2x+5)+\tan^{-1}\left(\frac{x-1}{2}\right)\right)+C.$$

Okay, so we can integrate rational functions whenever the denominator factors into a product of (different) linear terms and when it is an irreducible quadratic via trigonometric substitution. What about more complicated polynomials?

It turns out that this is really all there is: every polynomial over the real numbers factors as a product of linear and quadratic terms. This is not too hard to see using complex numbers, and we may talk about it a bit later in the semester, but for now just take my word for it. This means that given any polynomial in the denominator, we can factor it into linear and quadratic factors, which we can use the method of partial fractions to separate into integrals of the form $\frac{A}{?}$ where A is a constant and ? is either a linear polynomial or an irreducible quadratic, and we know how to compute either of these kinds of integrals.

The only remaining difficulty is when the denominator has repeated factors. For example, consider

$$\int \frac{2x-1}{(x-1)^2}\,dx.$$

We can't separate this in the usual way, because we would end up with both terms of the form $\frac{A}{x-1}$, and A cannot be a constant (this would mean that $x-1$ divides $2x-1$ as a polynomial, which is not true).

Since we may have constants we can't control in the numerator, we should expect that in addition to a term like $\frac{A}{x-1}$ there may also be a term of the form $\frac{B}{(x-1)^2}$. Indeed, this turns out to solve the problem: if

$$\frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2},$$

then multiplying through by $(x-1)^2$ gives

$$2x-1 = A(x-1) + B = Ax + B - A,$$

which we can use to see directly that we must have $A = 2$ and so $B = 1$. Alternatively, we could use a similar approach to above: plugging in $x = 1$ gives $1 = B$, and then $A = \frac{2x-1-B}{x-1} = \frac{2x-2}{x-1} = 2$. Therefore we find

$$\int \frac{2x-1}{(x-1)^2}\,dx = 2\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx = 2\log(x-1) - \frac{1}{x-1} + C.$$

We can use this kind of method to compute rational functions of trigonometric functions, extending last class's results. For example, consider

$$\int \frac{1}{\tan^2 x - 1}\,dx.$$

This has the opposite sign we usually like to see on $\tan^2 x$, so it's not convenient for trigonometric identities. Instead, we can substitute $u = \tan x$, so $du = \sec^2 x\,dx = (\tan^2 x + 1)\,dx = (u^2+1)\,dx$, to get

$$\int \frac{1}{\tan^2 x - 1}\,dx = \int \frac{1}{u^2-1} \cdot \frac{1}{(u^2+1)}\,du = \int \frac{1}{u^4-1}\,dx.$$

Computing this integral is on your homework, and is possible through the method of partial fractions.