Lecture 4: trigonometric substitution

Calculus II, section 3
January 31, 2022

So far, the primary methods of integration we know are u-substitution and integration by parts. Today, we're going to use those to create more methods: first, we'll use integration by parts (among other tools) to compute some trigonometric integrals, and once we know how to do those we'll see how we can use clever substitutions to transform more complicated integrals into trigonometric integrals.

1. Trigonometric integrals

Consider the integral $\int \sin^4(x)\cos(x)\,dx$. This is amenable to substitution: if $u = \sin x$, then $du = \cos(x)\,dx$ and so our integral is $\int u^4\,du = \frac{1}{5}u^5 = \frac{1}{5}\sin^5(x)$.

How about $\int \sin^4(x)\cos^3(x)\,dx$? If we try to do the above method directly, we get stuck: there's extra factors of $\cos(x)$. But if we notice that in fact there are exactly two extra factors, we can use the equation $\cos^2(x) = 1 - \sin^2(x)$: in other words, write

$$\begin{aligned}\int \sin^4(x)\cos^3(x)\,dx &= \int \sin^4(x)\cos^2(x)\cdot\cos(x)\,dx \\ &= \int \sin^4(x)(1-\sin^2(x))\cdot\cos(x)\,dx \\ &= \int u^4(1-u^2)\,du\end{aligned}$$

where $u = \sin x$. This is an integral we can solve, since it is a polynomial:

$$\int u^4(1-u^2)\,du = \int u^4 - u^6\,du = \frac{1}{5}u^5 - \frac{1}{7}u^7 = \frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x).$$

If we replaced $\cos^3(x)$ by $\cos^5(x)$ or $\cos^7(x)$ we could essentially do the same thing, i.e. peel off one factor of $\cos x$ and replace the remaining term by some power of $1 - \sin^2(x)$. Similarly, if we have something like $\int \sin^3(x)\cos^6(x)\,dx$ we can do the same method, switching the role of sine and cosine.

If both sine and cosine have even exponents, though, this doesn't work. For example, how can we evaluate the integral $\int \cos^4(x)\,dx$? There are no $\sin(x)$ terms for the substitution $u = \cos x$, and $u = \sin x$ doesn't get us anywhere either.

Instead, we can integrate by parts. Set $u = \cos^3(x)$ and $dv = \cos(x)\,dx$, so that $du = -3\cos^2(x)\sin(x)$ (by the chain rule) and $v = \sin(x)$. Then

$$\int \cos^4(x)\,dx = \int u\,dv = uv - \int v\,du = \cos^3(x)\sin(x) + \int 3\cos^2(x)\sin^2(x)\,dx.$$

This is no easier an integral, but now we can apply our trick of replacing $\sin^2(x)$ by $1-\cos^2(x)$ to get back to something more like our original integral:

$$\int \cos^4(x)\,dx = \cos^3(x)\sin(x) + 3\int \cos^2(x)\,dx - 3\int \cos^4(x)\,dx,$$

so solving for the original integral we get

$$\int \cos^4(x)\,dx = \frac{1}{4}\left(\cos^3(x)\sin(x) + 3\int \cos^2(x)\,dx\right).$$

Thus we've reduced the question of integrating $\cos^4(x)$ to that of integrating $\cos^2(x)$. This is often how integrals like this will go: rather than giving us a solution immediately, this process will reduce us to another (hopefully easier) integral, and we can repeat until we get to something that we know.

In this case, we actually do know the integral of $\cos^2(x)$ from last class, where we computed it via essentially exactly this method: it is $\frac{1}{2}(\sin(x)\cos(x) + x)$ (up to an additive constant). Therefore in all we have

$$\int \cos^4(x)\,dx = \frac{1}{8}(2\sin(x)\cos^3(x) + 3\sin(x)\cos(x) + 3x) + C.$$

By doing variants of this method, we see that we can compute any integral of the form $\int \sin^m(x)\cos^n(x)\,dx$, where m and n are integers; this is hardest when m and n are both even. It's also possible to use the double angle formula instead of integration by parts; a problem walking you through this method will be on your homework for this week.

It turns out that (with some modifications) this method works when m and/or n are negative, too, so we can also compute integrals of ratios of sines and cosines and their inverses (i.e. tangents, secants, and cosecants). In this case, because we are sometimes integrating functions of the form $\frac{1}{\text{something}}$ we often end up with logarithms in the final answer, unlike the case with positive exponents.

For example, let's integrate $\tan x$. We have

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx,$$

so if $u = \cos x$ so that $du = -\sin x\,dx$ we have

$$\int \tan x\,dx = -\int \frac{1}{u}\,du = -\log u = -\log(\cos x)$$

up to a constant. Sometimes we may want to evaluate this when $\cos x$ is negative, e.g. $\int_{2\pi/3}^{4\pi/3} \tan x\,dx$; in this case we can substitute $u = -\cos x$ so that $du = \sin x\,dx$, so in this case the antiderivative is, up to a constant,

$$\int \tan x\,dx = \int \frac{1}{-u} = -\log u = -\log(-\cos x)$$

In other words when $\cos x$ is negative we can replace it by $-\cos x$, so that in general the antiderivative of $\tan x$ is given by $-\log(|\cos x|)+C$, which is defined more broadly. Note that this is still undefined when $\cos x = 0$, i.e. at $x = \pi + 2\pi n$ for every integer n; this makes sense because $\tan x$ is also undefined at such x. We can similarly compute $\int \sec x \, dx = \int \frac{1}{\cos x} \, dx$, though we need a clever substitution to cancel terms, namely $u = \sec x + \tan x$ so that $du = \sec x(\sec x + \tan x) \, dx = u \sec x \, dx$ (by the chain rule) and therefore

$$\int \sec x \, dx = \int \frac{1}{u} \, du = \log u = \log(\sec x + \tan x),$$

which similarly becomes $\log|\sec x + \tan x|$.

2. Trigonometric substitution

We've talked about the areas and volumes of some geometric shapes before, but one we've skipped over is one of the simpler ones, namely the area of a disk (i.e. the area enclosed by a circle). Let's do that now, say just with a disk of radius 1 for simplicity. The equation of a circle of radius 1 is just $x^2 + y^2 = 1$, so we are looking for the area bounded by the equations $y = \pm(1 - x^2)$. To reduce to a single integral, we can make use of the vertical symmetry: the area of a disk is twice the area of the upper half-disk, which is $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$. In fact, using the horizontal symmetry we can simplify even further: this is in turn twice the area of a quarter-disk, which is given by $\int_0^1 \sqrt{1 - x^2} \, dx$. How can we compute this integral?

The obvious substitution is $u = 1 - x^2$, but this doesn't do much for us since there's no x term outside the square root. Keeping in mind what we've learned, namely that trigonometric integrals are generally computable, let's try and make a substitution that turns this into a trigonometric integral. Instead of writing u as a function of x, in this case it's more convenient to write x as a function of some other variable θ; let's say $x = \sin\theta$. (Of course, this is the same thing as setting $\theta = \cos^{-1} x$, since $\cos^{-1}$ is well-defined on this interval.)

In this case, we have $dx = \cos\theta \, d\theta$ and so

$$\int \sqrt{1 - x^2} \, dx = \int \cos\theta \cdot \sqrt{1 - \sin^2\theta} \, d\theta.$$

Since $1 - \sin^2\theta = \cos^2\theta$ and $\cos\theta$ is positive in the first quadrant, we can safely write this as $\int \cos^2\theta \, d\theta$, which we've already computed to be $\frac{1}{2}(\sin\theta\cos\theta + \theta)$. To switch back to x, we can replace $\sin\theta$ by x, but how can we deal with $\cos\theta$ and θ? Well, we can use the equation $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}$ again in the first case, and for the θ term simply write $\theta = \sin^{-1}(x)$. Therefore we have computed the antiderivative

$$\int \sqrt{1 - x^2} \, dx = \frac{1}{2}\left(x\sqrt{1 - x^2} + \sin^{-1}(x)\right) + C,$$

valid on the interval from 0 to 1. Evaluating, we conclude that

$$\int_0^1 \sqrt{1 - x^2} \, dx = \frac{1}{2}(0 + \sin^{-1}(1)) - \frac{1}{2}(0 + \sin^{-1}(0)) = \frac{1}{2}(\sin^{-1}(1) - \sin^{-1}(0)),$$

so the area of a disk of radius 1 is $2(\sin^{-1}(1) - \sin^{-1}(0))$. If we take our interval for θ to be, say, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (which contains exactly the full range of values of $\sin\theta$), then $\sin^{-1}(0) = 0$ and $\sin^{-1}(1) = \frac{\pi}{2}$, so we conclude that the area of our disk is π, which agrees with the usual formula πr^2 at $r = 1$.

The general principle here is: if we're integrating something with a term that reminds us of a trigonometric identity, try substituting x for a trigonometric function and see if we can make use of this identity.

Another example is antidifferentiating $\frac{1}{\sqrt{1-x^2}}$. If we again set $x = \sin\theta$, we have

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{1}{\cos\theta}\cos\theta\,d\theta = \theta + C = \sin^{-1}(x) + C.$$

Before seeing this integral it isn't even obvious what the derivative of $\sin^{-1}(x)$ is! (Of course, this is only defined for x between -1 and 1; but the same is true of the original integral.)

If we instead chose to substitute $x = \cos\theta$, we would get $dx = -\sin\theta\,d\theta$ and so

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-\cos^2\theta}}\sin\theta\,d\theta = -\int \frac{1}{\sin\theta}\sin\theta\,d\theta = -\theta + C = -\cos^{-1}(x) + C.$$

Therefore there exists some constant C such that $\sin^{-1}(x) = -\cos^{-1}(x) + C$ for every x; we can find this constant by evaluating at 0, where we get $\sin^{-1}(0) = 1$ and $\cos^{-1}(0) = \frac{\pi}{2}$, so $C = \frac{\pi}{2}$. Therefore we have $\sin^{-1}(x) = \frac{\pi}{2} - \cos^{-1}(x)$ for every x, which is a reflection of the identity $\sin(\theta) = \cos(\frac{\pi}{2} - \theta)$.

Beyond this sort of fun, this is a demonstration of a more general principle: when doing trigonometric substitutions, $\cos\theta$ and $\sin\theta$ are usually more or less equivalent, and you should feel free to choose whichever is more convenient. (This is similar to the idea of integrating in different directions to get a more convenient formula.)

What about similar but more complicated versions? For example, say we have $\int \frac{1}{\sqrt{4-x^2}}\,dx$. This is very similar to the previous version, but now there is no special property of $\sqrt{4-\sin^2\theta}$. What can we do instead?

Well, we want something that will cancel out that 4: instead of $x = \sin\theta$, try $x = 2\sin\theta$. Then we have $dx = 2\cos\theta$ and

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,d\theta = \int \frac{1}{2\cos\theta}2\cos\theta\,d\theta = \theta + C = \sin^{-1}\left(\frac{x}{2}\right).$$

This generalizes in the way you'd expect; there will be a problem on this week's homework going over such generalizations.

Once we understand this sort of integral, the first change that comes to mind to make is to flip the sign–can we compute for example $\int_4^8 \frac{1}{\sqrt{x^2-16}}\,dx$?

Along the same lines as last time, we could substitute $x = 4\sin\theta$, so that the (indefinite) integral becomes

$$\int \frac{1}{\sqrt{16\sin^2\theta - 16}}4\cos\theta\,d\theta = \int \frac{1}{4\sqrt{-\cos^2\theta}}4\cos\theta\,d\theta.$$

This is a bit of a problem!

Instead, we want to try a different trigonometric substitution: in this case $x = 4\sec\theta = \frac{4}{\cos\theta}$. The key property of the secant is that it has the relationship $\sec^2\theta = 1 + \tan^2\theta$, as can be seen by dividing $1 = \cos^2\theta + \sin^2\theta$ by $\cos^2\theta$. In particular this means that $\sec^2\theta - 1 = \tan^2\theta$, and so if $x = 4\sec\theta$ then $\sqrt{x^2-16} = \sqrt{16(\sec^2\theta - 1)} = 4\tan\theta$. We also have $\frac{d}{d\theta}\sec\theta = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$ by the chain (or division) rule, so $dx = 4\tan\theta\sec\theta$ and thus

$$\int \frac{1}{\sqrt{x^2-16}}\,dx = \int \frac{4\tan\theta\sec\theta}{4\tan\theta}\,d\theta = \int \sec\theta\,d\theta.$$

To conclude, we can recall this integral from the last section to get

$$\int \frac{1}{\sqrt{x^2-16}} = \log|\sec\theta + \tan\theta| + C = \log\left|\frac{x}{4} + \frac{1}{4}\sqrt{x^2-16}\right| + C.$$

Thus evaluating we have

$$\begin{aligned}\int_4^8 \frac{1}{\sqrt{x^2-16}}\,dx &= \log\left(\frac{8}{4} + \frac{1}{4}\sqrt{64-16}\right) - \log\left(\frac{4}{4} + \frac{1}{4}\sqrt{16-16}\right)\\ &= \log(2+4\sqrt{3}) - \log 1\\ &= \log(2+\sqrt{3}).\end{aligned}$$

Since secant and tangent have appeared as a pair like sine and cosine, we might wonder whether similarly they are equivalent for these kinds of substitutions. However, the asymmetry of the relationship between them suggests that they are not: we have $\sec^2\theta - 1 = \tan^2\theta$ for going from secant to tangent (i.e. $\sec\theta$ substitutions), but $1 + \tan^2\theta = \sec^2\theta$ for going the other way. This suggests we should look for $\tan\theta$ substitutions when we have things of the form $x^2 + 1$ (or with other constants).

We can compute the basic example for this, which goes as above. Let's try something slightly different-looking: dropping the square root. This is actually even more convenient to integrate in this case: if $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$ and so

$$\int \frac{1}{x^2+1}\,dx = \int \frac{1}{\tan^2\theta + 1}\sec^2\theta\,d\theta = \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta = \theta + C = \tan^{-1}(x) + C.$$

Note that, like $\frac{1}{x^2+1}$, the inverse tangent function is defined on all real numbers, unlike for the inverses of most other trigonometric functions.