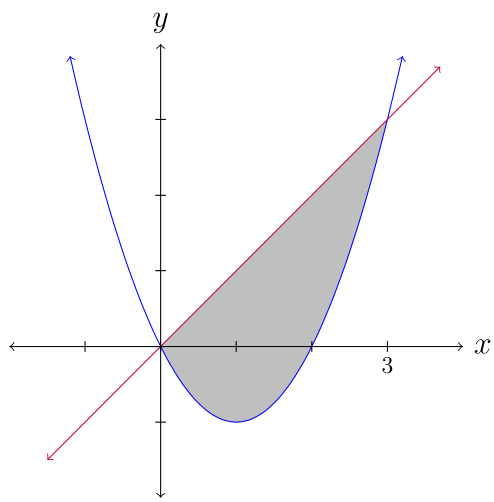
Lecture 2: integrals and volume

Calculus II, section 3
January 24, 2022

Let's briefly recall what an integral is: we want to find some cumulative area under the curve given by the graph of some function. Sometimes we can do this geometrically or via a limit definition, but usually we want to apply the fundamental theorem of calculus to turn this into a question of finding antiderivatives, which we can do in a couple of ways: either recognizing our function as the derivative of something we know, u-substitution, or some combination (or various other tools we will learn).

Today, we'll apply integrals to some more geometric problems, and look at different ways we can use integrals to solve the same problem in the easiest way.

The most straightforward generalization of the idea of the integral is: instead of asking for the area under a curve, i.e. the area between the curve and the x-axis, let's ask for the area between two curves. For example, consider the area between the curves given by $y = x^2 - 2x$ and $y = x$:



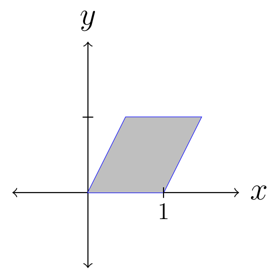
The height of the gray area at x is the difference between the two functions, namely $x-(x^2-2x) = 3x-x^2$. The endpoints are at the intersection of the curves, where $x = x^2-2x$, i.e. $x^2 = 3x$, which occurs at $x = 0$ and $x = 3$. Therefore the total area of this shape is

$$\int_0^3 3x - x^2\,dx.$$

By the fundamental theorem of calculus, this is the antiderivative of $3x-x^2$, namely $\frac{3}{2}x^2-\frac{1}{3}x^3$, evaluated at 3 and 0:

$$\left(\frac{3}{2}\cdot 3^2 - \frac{1}{3}\cdot 3^3\right) - \left(\frac{3}{2}\cdot 0^2 - \frac{1}{3}\cdot 0^3\right) = \frac{27}{2} - 9 = \frac{9}{2}.$$

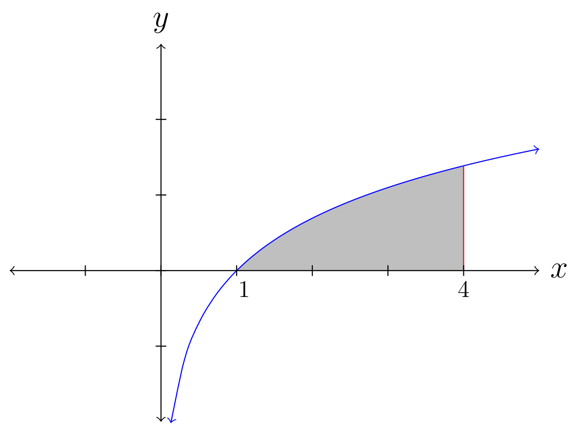
Let's think about computing the area of a simple shape, a parallelogram.



We know from geometry that the area of a parallelogram is length times height, so $1 \cdot 1 = 1$ in this case. But if we were to treat this as an integral, we would need to divide it into three different sections: from $x = 0$ to $x = \frac{1}{2}$, it is the area between $y = 2x$ and $y = 0$; from $x = \frac{1}{2}$ to $x = 0$, it's the area between $y = 1$ and $y = 0$; and from $x = 1$ to $x = \frac{3}{2}$, it's the area between $y = 1$ and $y = 2x - 2$. This seems like a massive pain in order to do something simple.

We can fix this by thinking of this integral differently: instead of thinking about cumulative height as we go horizontally, think about cumulative width as we go vertically. In other words, we treat y as the independent variable. In this case, this is all one integral: it is the area between $x = \frac{1}{2}y$ and $x = \frac{1}{2}y + 1$ from $y = 0$ to $y = 1$, i.e. $\int_0^1 1\,dy = 1$.

Another example is integrating $\log x$. Since $y = \log x$ is the same thing as $x = e^y$, we can often choose which to use, and since $\frac{d}{dy}e^y = e^y$ the latter is quite easy (although we can also integrate $\log x$ directly). For example, $\int_1^4 \log x\,dx$ is the area of the following shape:



If we integrate this with respect to x, we need to know the antiderivative of the logarithm, which we can find but is somewhat nontrivial. On the other hand if we integrate with respect to y, this is the area between $x = 4$ and $x = e^y$ from $y = 0$ to $y = \log 4$, i.e.

$$\int_0^{\log 4} 4 - e^y\,dy = (4 \cdot \log 4 - e^{\log 4}) - (4 \cdot 0 - e^0) = 4 \log 4 - 3.$$

In fact, we can do this more generally: instead of taking the integral from 1 to 4, let's take it from 1 to z (since we're already using x and y). Then we have

$$\int_1^z \log x\,dx = \int_0^{\log z} z - e^y, dy = (z \log z - e^{\log z}) - (0 - 1) = z \log z - z + 1.$$

By the fundamental theorem of calculus, if $F(x)$ is an antiderivative of $\log x$ then

$$\int_1^z \log x\,dx = F(z) - F(1),$$

so we've proven that any antiderivative of $\log x$ is of the form $x\log x - x$ plus some constant $1 + F(1)$. We've found a whole new method to integrate $\log x$!

Our next application is to volumes. There is of course a whole subject about doing calculus in higher dimensions, but we don't need that much theory: in the same way that we think of area as cumulative height (or distance), we can think of volume as cumulative area. For example, consider a cylinder of radius r and height h. Each "layer" of the cylinder has area πr^2, so the total volume is the integral of those areas, i.e.

$$V = \int_0^h \pi r^2\,dx = \pi r^2 h.$$

Of course, we already know this formula from geometry.

Another formula we may recall from geometry is the volume of a cone with radius r and height h: $V = \frac{1}{3}\pi r^2 h$. This formula is a little more mysterious; let's see if we can figure out where it comes from.

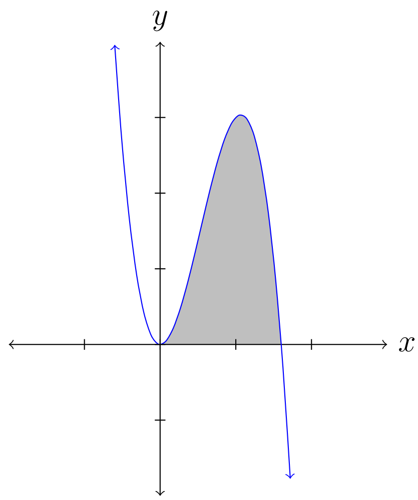
At the base, the bottom layer of the cone has radius r, and so area πr^2. As we go up, though, the radius of each layer shrinks until we get to the top layer, which is just a point with area 0. Since the profile of the cone is linear, the change in radius is linear, and so the fact that it is r at the bottom and 0 at the top lets us find the formula: the radius of the layer at height x is $(1 - x/h)r$, so the area of the layer at height x is $\pi(1 - x/h)^2 r^2$. Integrating, we get that the total volume of the cone is

$$\int_0^h \pi(1 - x/h)^2 r^2 = \int_0^h \pi u^2 r^2(-h)\,du = -\pi r^2 h \cdot \frac{u^3}{3}\bigg|_0^h = \frac{1}{3}\pi r^2 h(1-0) = \frac{1}{3}\pi r^2 h,$$

where we substitute $u = 1 - x/h$.

Notice that we could have made things slightly easier for ourselves by starting at the top of the cone rather than the bottom: then the layer at height x down from the bottom instead of up from the top has radius $\frac{x}{h}r$, and so area $\pi\frac{x^2}{h^2}r^2$ so that the integral is much simpler and we don't have to substitute.

Again, we can switch the direction in which we do this. Consider the solid formed as follows: take the region bounded by the x-axis and the curve $y = 8x^2 - 5x^3$, and rotate it about the y-axis.



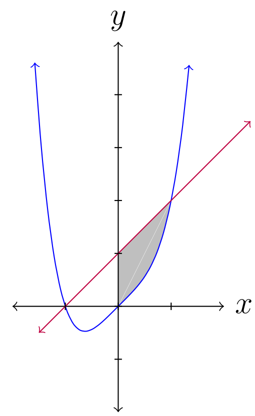
To compute the volume of the resulting solid, we could use the method we used on the cone, but this is somewhat difficult since we don't know how to easily estimate the radius of each layer. We could also use methods particular to solids of rotation, which we'll talk about more when we cover polar coordinates, but for now this isn't something we're comfortable with and in any case we'd like a simpler technique.

One thing we can do is very similar to the previous technique, but in a "different direction": again compute volume as cumulative area, but now of a cylindrical shell. That is: each layer is the surface of a cylinder (minus top and bottom), and as the radius varies so does the height and thus the total area; integrating this area over all possible radii gives the total volume.

In our example, at radius r the height of the cylindrical shell is $8r^2 - 5r^3$. The area of each cylinder of radius r and height h is itself an integral over its height of the length of each layer, i.e. $2\pi r$, for a total area of $2\pi rh$; thus each shell in our case has area $2\pi r(8r^2 - 5r^3)$. Since the largest radius in our solid is at $r = \frac{8}{5}$, we have in total

$$V = \int_0^{8/5} 2\pi r(8r^2 - 5r^3)\,dr = 2\pi \int_0^{8/5} 8r^3 - 5r^4\,dr = 2\pi\left(2\cdot\left(\frac{8}{5}\right)^4 - \left(\frac{8}{5}\right)^5\right) = \frac{1}{2}\left(\frac{8}{5}\right)^5 \pi.$$

For another example, let's take the "cup" shape formed by rotating the area between the curves $y = x + 1$ and $y = x^4 + x$.



The height of the cylindrical shell at radius r is the height of the gray area, i.e. $(r+1)-(r^4+r)=1-r^4$, so each shell has area $2\pi r(1-r^4)$. Therefore the volume of this shape is

$$V=\int_0^1 2\pi r(1-r^4)\,dr=2\pi\int_0^1 r-r^5\,dr=2\pi\left(\frac{1}{2}-\frac{1}{6}\right)=\frac{2}{3}\pi.$$