Lecture 18: Taylor series

Calculus II, section 3
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Last time, we looked at power series, and found that by integrating the geometric series we got

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

and so

$$\log x = \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n}.$$

The question we're interested in today is: how do we generalize this? That is, given some function of interest $f(x)$, how do we write it as a power series?

Let's start with the "standard" power series about the origin, so really what our question boils down to is finding a sequence a_n such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for every x, or at least for every x for which the series on the right-hand side converges. (For example, $\frac{1}{1-x}$ is defined for all $x \neq 1$, but the corresponding geometric series $\sum_{n=0}^{\infty} x^n$ is defined only for $|x| < 1$.)

Let's assume that we have such an expansion

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

How can we extract a formula for the a_n in terms of f?

Well, for a_0, this is easy: $f(0) = a_0$, since it is the constant term. The other terms vanish after that, though, so we can't use the same method to get them.

One thing we could try is subtracting a_0: then we have

$$f(x) - a_0 = a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

which has a factor of x which we could pull out. Thus

$$\frac{f(x) - a_0}{x} = a_1 + a_2 x + a_3 x^2 + \cdots,$$

and so evaluating the right-hand side at $x = 0$ gives a_1. On the left-hand side, we can't plug in $x = 0$ since x is in the denominator, but we can take the limit; since $a_0 = f(0)$, this limit has another name, i.e.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0).$$

Thus $a_1 = f'(0)$.

We might then guess that $a_2 = f''(0)$ is the second derivative, and so on. This isn't quite true, as we can see by differentiating the whole power series term by term. We have

$$f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1},$$

so the constant term is at $n = 1$ and therefore $a_1 \cdot 1 = a_1$; the second derivative is

$$f''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2},$$

and so the constant term is at $n = 2$, namely $a_2 \cdot 2 \cdot 1 = 2a_2 = f''(0)$. Therefore $a_2 = \frac{f''(0)}{2}$.

To continue this pattern, to make a_n the constant term we want to differentiate n times. The nth derivative of $a_n x^n$ is $a_n n(n-1)(n-2)\cdots 2 \cdot 1 \cdot x^0 = a_n n!$, and so we conclude that a_n is the nth derivative of f evaluated at $x = 0$, written $f^{(n)}(0)$. Thus if $f(x)$ can be written as a power series in some region around the origin, then it is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This is Taylor's formula, and the series on the right is the Taylor series for $f(x)$.

This is a really remarkable formula. At least under reasonable conditions, what this says is that by studying the local behavior of a function at a single point—namely all of its higher derivatives—we can get an exact formula for this function on some larger range, and for many functions everywhere in its domain!

Let's do an example we've looked at before: the exponential function e^x. The nth derivative of e^x will always be e^x, so $f^{(n)}(0) = e^0 = 1$ for every n. Therefore Taylor's formula says that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We could check this by differentiating the right-hand side and seeing that the result is equal to the original series. This power series has an infinite radius of convergence, so it gives the exponential function for all real (or indeed complex) numbers.

We could also take this to be the definition of the exponential function, and it's possible (though annoying) to work out from the series definition all of the usual properties (product of exponentials, power rule, etc.). It's also useful for approximations because it converges so fast: taking just the first-order approximation, $e^x \approx 1 + x$ for small x, is already a useful heuristic for many purposes.

Another simple example is polynomials: polynomials will always be equal to their Taylor series, since they are a certain simple kind of power series. For example, if $f(x) = x^2 - x$, then $f'(x) = 2x - 1$, $f''(x) = 2$, and all higher derivatives are 0, so $f(0) = 0$, $f'(0) = -1$, and $f''(0) = 2$ so the Taylor series is

$$0 - x + \frac{2}{2}x^2 = x^2 - x.$$

A slightly more complicated example is given by the sine and cosine functions. The key property of these functions, at least from the perspective of differential equations, is that differentiating twice recovers the negative of the original function, i.e. they solve the differential equation $y'' = -y$. We also know that derivatives of sine and cosine map to each other, so it's relatively easy to write down all their derivatives, though a little more work than for the exponential function: the derivatives of $\sin x$ (starting from the 0th derivative, i.e. the original function) are $\sin x$, $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, $\dots$. These are all easy to evaluate at 0, and give 0, 1, 0, -1, 0, 1, $\dots$ respectively. In particular $\sin x$ will only have terms coming from the odd n; this is because it is an odd function, i.e. $\sin(-x) = -\sin x$, which is true of odd powers but not even powers. Similarly $\cos x$ has the same chain of derivatives, just shifted, and evaluated at 0 will give $1, 0, -1, 0, 1, 0, \dots$, with all terms coming from even n, as it is an even function, i.e. $\cos(-x) = \cos x$.

Thus we want to write the sine and cosine functions in terms of odd and even indices:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}, \qquad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}.$$

Taking the first-order approximation of $\sin x$ gives the small-angle formula $\sin x \approx x$ for small x; doing the same for $\cos x$ gives $\cos x \approx 1$ since the first-order term is 0. If a better approximation is needed, we could add the second term: $\cos x \approx 1 - \frac{1}{2}x^2$.

We can now actually prove Euler's formula! Euler's formula gives us a way to decompose e^{ix} in terms of a real term and an imaginary term. Well, now that we have a power series representation of e^x, we can just plug in ix:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} - \cdots.$$

Note that the even terms are all real, since if $n = 2k$ then $i^n = i^{2k} = (i^2)^k = (-1)^k$, while the odd terms are all imaginary, since if $n = 2k+1$ then $i^n = i^{2k+1} = (-1)^k i$. If we took just the even terms, we would get

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k};$$

this is just the Taylor series for $\cos x$! If we take the odd terms and pull out a factor of i, we get

$$i\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1},$$

which is i times the Taylor series for $\sin x$, and so we conclude that

$$e^{ix} = \cos x + i \sin x.$$

All of these examples have infinite radius of convergence, so they make sense for all real numbers. This is not always the case; an easy example is the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Taylor's theorem tells us that the nth derivative of $\frac{1}{1-x}$ evaluated at 0 must be equal to $n!$, and indeed this is not too hard to check:

$$\frac{d^n}{dx^n}\frac{1}{1-x} = \frac{n!}{(1-x)^{n+1}}.$$

But this converges only for $|x| < 1$, so it does not define $\frac{1}{1-x}$ on the whole real line but only on part of it. What if we wanted to understand $\frac{1}{1-x}$ as a power series near some point b with $|b| > 1$?

Well, we've looked at such power series too, so we can ask the generalized question: how can we write a function $f(x)$ as a power series

$$\sum_{n=0}^{\infty} a_n (x-b)^n \quad ?$$

We can do essentially the same thing as above, except that we replace 0 everywhere by b—after all, in this setting there's nothing special about 0, it's just some arbitrary point on the line, and b is just as good. Thus $f(b) = a_0$, $f'(b) = a_1$, $f''(b) = 2a_2$, and so on:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

This is the general Taylor series formula; the special case above where $b = 0$ is often called the Maclaurin series.

For example, we could not have written the logarithm as a Maclaurin series, because it is not even defined at $x = 0$. Instead, we can expand it about $x = 1$, and we get the same series as last time: the derivatives of $\log x$ are $\frac{1}{x}$, $-\frac{1}{x^2}$, $\frac{2}{x^3}$, $-\frac{6}{x^4}$, etc., and it is not hard to verify that the nth derivative is $\frac{(n-1)!}{x^n}$ with value at $x = 1$ given by $(-1)^n(n-1)!$ for $n > 0$, with $\log 1 = 0$ for the 0th derivative. Therefore the Taylor expansion about $x = 1$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n$$

as last time. Note that this does *not* have infinite radius of convergence, as we discussed, so it only defines the logarithm in this region.

All of these are examples of functions which are (locally) *analytic* near the chosen base point (in the examples above, either 0 or 1, but in general potentially any point): they are infinitely differentiable, so it is possible to write down the Taylor series; this series converges, at least for x sufficiently close to the base point, i.e. the power series has a nonzero radius of convergence; and where it does converge, it is actually equal to the original function. These are the functions for which this miracle of looking at entirely local information (the derivatives) to extract a global formula (the Taylor series) is possible. This is not always true.

It's not hard to find examples of functions which are not infinitely differentiable—for example, $f(x) = |x|$. A little trickier is to find a function which is infinitely differentiable (also called smooth) but not analytic. One example is given by the bump function, defined to be $e^{-\frac{1}{1-x^2}}$ for $|x| < 1$ and 0 otherwise; this turns out to be smooth, but it's easy to see it cannot be analytic despite having no poles, since for $|x| > 1$ all the derivatives are 0 but the function is not everywhere 0.

In the case of complex functions, this issue disappears. We talked before about how limits existing in the complex setting is a much more powerful condition, and in particular differentiability is much stronger: in fact, if a complex function $f(z)$ is differentiable, then it is automatically smooth and locally analytic, i.e. for every point b at which $f(z)$ is analytic it has a Taylor expansion of some radius (which might be arbitrarily small, or might be infinite) about b. In the complex setting this is usually called being holomorphic at b, and functions are said to be holomorphic if they are holomorphic everywhere, i.e. are differentiable (or equivalently have a Taylor expansion) on the entire complex plane. Most interesting complex functions are either holomorphic or *meromorphic*, i.e. the ratio of two holomorphic functions, generalizing the notion of rational functions as the ratio of polynomials.