Lecture 17: power series

Calculus II, section 3
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Recall from last time we started to look at power series, i.e. series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

for some sequence a_n and number x, which we can think of as a variable.

The first question to ask about such things is: when do they converge? For example, one of the simplest power series is the geometric series

$$\sum_{n=0}^{\infty} x^n,$$

with $a_n = 1$ for every n, which we know converges when $|x| < 1$ and diverges when $|x| > 1$. In general, we'd like to have a simple criterion based on x to say when the series converges or diverges.

How could we determine whether

$$\sum_{n=0}^{\infty} a_n x^n$$

converges? Well, by applying our convergence tests! Let's try the root test: the series converges if

$$\lim_{n\to\infty} |a_n x^n|^{1/n} = |x| \lim_{n\to\infty} |a_n|^{1/n}$$

converges to something less than 1, or equivalently if

$$\lim_{n\to\infty} |a_n|^{1/n}$$

converges to something less than $\frac{1}{|x|}$ and diverges if it converges to something greater than $\frac{1}{|x|}$. Thus if this limit converges to some number, say L, then for a given x the series converges if $|x| < \frac{1}{L}$ and diverges if $|x| > \frac{1}{L}$. In this case $r := \frac{1}{L}$ is called the radius of convergence of the power series. For example, the geometric series has radius of convergence 1. Note that it may happen that $L = 0$; in this case we say that the power series has infinite radius of convergence, i.e. it converges for every x.

Here's another example: if $a_n = e^n$, what is the radius of convergence of the corresponding power series

$$\sum_{n=0}^{\infty} e^n x^n \ ?$$

Let's apply the root test again:

$$\lim_{n\to\infty} |e^n x^n|^{1/n} = \lim_{n\to\infty} e|x| = e|x|,$$

or more directly $L = e$ and so $r = \frac{1}{e}$, i.e. the series converges for $|x| < \frac{1}{e}$ and diverges for $|x| > e$.

We could also apply the ratio test: the series converges when

$$\lim_{n\to\infty} \left| \frac{e^{n+1}x^{n+1}}{e^n x^n} \right| = \lim_{n\to\infty} e|x| = e|x|$$

is less than 1, which is the same as above. In general, we could define the radius of convergence using the ratio test as well:

$$r = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Finally, this case is actually basically just a geometric series again: we could observe that $e^n x^n = (ex)^n$, and so this is the geometric series evaluated at ex. Therefore it converges if $|ex| < 1$ and diverges if $|ex| > 1$, so the radius of convergence is $\frac{1}{e}$.

Why is this called a radius of convergence? Well, it's establishing a number r such that the series converges if x is within distance r of the origin, which is a sort of radius. Still, the term "radius" suggests some sort of circle, and this doesn't look much like a circle (arguably a zero-dimensional one, but otherwise). We can fix this by expanding to the complex plane: there, the set of x for which a power series with radius of convergence r converges is a disk of radius r about the origin. We won't deal much with complex numbers in this part of the class (though they'll come up occasionally), but it's useful for this interpretation.

This suggests a generalization. When we talk about circles, we can control two things: the radius and the center. Here, we only have the radius: can we change the center as well?

Yes! Let a be any number; then we can look at the power series centered at a

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots.$$

When does this converge?

We can do the same sort of thing: applying the root test shows that this converges when

$$\lim_{n\to\infty} |a_n(x-a)^n|^{1/n} = |x-a| \lim_{n\to\infty} |a_n|^{1/n}$$

is less than 1, so the radius of convergence is defined by the same formula

$$r = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$$

but now the series converges if $|x-a| < r$ (and diverges if $|x-a| > r$), i.e. converges when x is within distance r of the point a. This corresponds to the circle of radius r about a.

For example, consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}.$$

The radius of convergence is

$$\lim_{n\to\infty} \frac{1}{|n|^{1/n}} = 1,$$

so the series converges for x within distance 1 of $a = 1$, i.e. for $0 < x < 2$, and diverges for $x < 0$ or $x > 2$.

What happens on the margin, i.e. for $|x - a| = r$? In this case the ratio and root test tell us nothing. In fact it can go either way, as this example shows: at $x = 2$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series which we know diverges. For $x = 0$, though, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which was our example of a conditionally convergent series!

Since on the interior of the disk of convergence the ratio and root tests tell us that the series converges absolutely, you might wonder if on the edges the series either diverges or converges conditionally. However it's also possible to converge absolutely on the edges: if we take the same series and replace the denominator by n^2, then we get the two series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which both converge, by the integral test or, in the second case, the alternating series test, and indeed converge absolutely.

Thus in total, given any power series around a center a, we can compute its radius of convergence r, and then the series will converge absolutely for $|x - a| < r$, diverge for $|x - a| > r$, and may converge absolutely, converge conditionally, or diverge on $|x - a| = r$. Fortunately in this last case there are only two things to check, i.e. $x = a + r$ and $x = a - r$.

Now that we understand convergence, i.e. treating the series as a sum for each possible choice of number x, let's switch to thinking of x as a variable. So long as the series converges absolutely, it is compatible with limits; this is a bit of a complicated statement which is really more in the domain of real analysis and I won't make you justify it, but for us the particular kind of limits we care about are differentiation and integration. In particular, if we have a power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

thinking of it as a function of x, we can differentiate term-by-term:

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} a_n n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} a_n n x^n = \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n.$$

We can antidifferentiate similarly (and thus integrate):

$$\int \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \int a_n x^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = C + x \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n = C + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n.$$

Depending on a_n, we may be able to understand this in terms of simpler functions. For example, we have

$$\frac{1}{x-1} = \sum_{n=0}^{\infty} x^n,$$

so

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \int \sum_{n=0}^{\infty} x^n \, dx = \int \frac{1}{1-x} \, dx + C.$$

By substituting $u = 1 - x$, so $du = -dx$, this integral is just

$$-\int \frac{1}{u} \, du = -\log u = -\log(1-x),$$

so for some constant C we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = C - \log(1-x).$$

For $x = 0$, the left-hand side is 0 and $-\log(1-0) = -\log 1 = 0$, so $C = 0$, and we have in general

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

This lets us come back to our example before:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

is given by plugging in $x - 1$ for x in this sum, and so

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = -\log(1-(x-1)) = -\log(2-x).$$

Before we found that at $x = 2$ this series diverges, which makes sense because $-\log(2-2) = -\log 0$ diverges, and at $x = 0$ it converges: in this case we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log(2-0) = -\log 2,$$

which I claimed last week.

In fact, this series allows us to write the logarithm as a power series in general: $\log x = -(-\log(1-(1-x)))$, so applying a negative sign and substituting $1-x$ gives

$$\log x = -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n}.$$

This is a power series centered at $x = 1$ with radius of convergence 1, so it defines the logarithm as a power series for $0 < x \leq 2$ but not for $x \leq 0$ (which makes sense) or $x > 2$ (which may be surprising). If you wanted to approximate the logarithm of some large number x, though, you could still do so using this series: using the power rule of logarithms, if we write $x = y^n$ for some large number n and $y \leq 2$ then we have $\log x = n \log y$, so we can use this series to compute $\log y$ and multiply by n to get the result. (We probably wouldn't use this in practice because the series converges pretty slowly, but for some functions exactly this sort of method is used—we'll talk about this more next week when we generalize this way of writing functions as power series via Taylor series.)