Lecture 15: first convergence tests

Calculus II, section 3
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Last time we introduced sequences and series, and in particular looked at some cases where we can see whether or not series converge and compute their value if so, such as geometric series. In particular we saw the following two tests:

- If $\lim_{n\to\infty} a_n$ fails to exist or exists and is nonzero, then $\sum_{n=1}^{\infty} a_n$ diverges. In other words, if $\sum_{n=1}^{\infty} a_n$ converges, then we must have $\lim_{n\to\infty} a_n = 0$.
- If we have two sequences a_n, b_n with $0 \leq a_n \leq b_n$ for every n, then

  $$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n,$$

  so if the sum of the b_n converges so does the sum of the a_n; conversely if the sum of the a_n diverges, so does the sum of the b_n.

The second of these, the comparison test, is only useful if we have something to compare it to; for this purpose the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1$$

is very useful. (We can also use it to show that series *diverge*: for example

$$\sum_{n=0}^{\infty} ne^n$$

diverges, since $ne^n \geq e^n$ for all $n \geq 1$ and $|e| > 1$. The finitely many terms where $ne^n < e^n$, i.e. just at $n = 0$, don't affect the convergence.)

We also last time looked at a trickier series $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which the above methods don't apply to but turns out to diverge due to a slick grouping argument. Today our goal is to introduce some more powerful methods which can handle this sort of thing.

For a_n positive, we can think of $\sum_{n=1}^{\infty} a_n$ as computing an area, by adding up the areas of $1 \times a_n$ rectangles over all n. This is reminiscent of the process for integrals, and indeed we can relate it to integrals: if $a_n = f(n)$ for some positive decreasing integrable function $f(x)$, then

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n).$$

On the other hand, we can shift everything by 1 to get a lower bound, and this does not affect the convergence of the series:

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)\,dx,$$

so the sum converges if and only if the integral does.

This lets us check the convergence of the harmonic series much more quickly. For the integral, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{N\to\infty} \log N - \log 1$$

which diverges, so

$$\sum_{n=1}^\infty \frac{1}{n}$$

must also diverge by the integral test.

More generally, this allows us to look at p-series, like for integrals: for which values of p does

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

converge?

We just saw that it does *not* converge for $p = 1$. More generally, we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \left(-\frac{1}{(p-1)x^{p-1}}\right)\Bigg|_1^\infty,$$

which converges (to $\frac{1}{p-1}$) for $p > 1$ and diverges for $p < 1$. Therefore the series also converges for $p > 1$ and diverges for $p \leq 1$. We've found a whole new family of series we can compute, which may be useful for the comparison test.

Since we know a lot of methods of integration, this is a very powerful test and can be applied to more complicated examples. For example, does

$$\sum_{n=1}^\infty \frac{n}{e^n}$$

converge or diverge?

Well, if we look at the corresponding integral

$$\int_1^\infty \frac{x}{e^x}\,dx,$$

this looks approachable: we can use integration by parts to see that the antiderivative is

$$-xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x},$$

which evaluated at 1 and N converges as $N \to \infty$. Therefore our series also converges.

Our methods so far generally require that a_n be nonnegative. If a_n may contain negative numbers, there's an easy way to get around this: replace it by its absolute value. Then it turns out that if

$$\sum_{n=1}^\infty |a_n|$$

converges, then so does

$$\sum_{n=1}^{\infty} a_n.$$

Why is this? Because of the comparison test! Namely, first observe that for any numbers $a_1, a_2, \ldots, a_n$, we have $|a_1 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$. This is because on the one hand if all the a_n have the same sign, i.e. are all positive or all negative, then the two sides are equal; if some of the signs are different, then there is some cancellation on the left-hand side making it smaller. Therefore

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|,$$

and the right-hand side is a sum of nonnegative terms, so if it converges to a finite number so must the left-hand side, which means that the original sum must converge.

In this case, where the sum of $|a_n|$ converges, we say that a_n converges absolutely; what we showed above is that if a_n converges absolutely, then it also converges in the regular sense. The converse is not true, however. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is an *alternating series*: the sign of the terms switches every time, i.e. it can be written as $(-1)^n b_n$ for some b_n which is always positive (or always negative), in this case $\frac{1}{n}$.

In this case, since $\frac{1}{n}$ is always getting smaller, the next term will always move back a little less than to where it was before, so the range it is moving between is getting smaller and smaller. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, eventually this range will get arbitrarily small; in other words this series will converge. (It turns out that it will converge to $-\log 2$.)

However, if we take absolute values we get

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, which we know does not converge! This series is convergent but not absolutely convergent; in this case we say it is conditionally convergent.

The way that we determined that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges generalizes. The key properties were that $\frac{(-1)^n}{n}$ is alternating; when we take the absolute value, in that case $\frac{1}{n}$, we get a sequence which is decreasing; and $\lim_{n\to\infty} \frac{(-1)^n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$. In general, if these properties hold then we can conclude that the series converges: i.e. if a_n is alternating, $|a_n|$ is decreasing, and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. (It may or may not converge absolutely.) This is called the alternating series test.

For example,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$$

is an alternating series; we have $\lim_{n\to\infty} \frac{(-1)^n}{(n+1)^2} = 0$ and $\frac{1}{(n+1)^2}$ is decreasing, so it satisfies the hypotheses of the alternating series test and therefore converges. In this case, it converges absolutely:

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

converges by the integral test. (We could make this simpler by reindexing, e.g. say $m = n+1$ so that the series is

$$\sum_{m=1}^{\infty} \frac{1}{m^2},$$

so we can more easily apply our knowledge of p-series.) Since any series which converges absolutely converges, we could actually have skipped the alternating series test in this case.

Alternating series have a weird property: their sum depends on the order. You're probably used to the idea that addition is commutative, though maybe not in that language: $a+b = b+a$, and the same is true when you add more numbers. For infinite series, though, that isn't necessarily true.

Consider for example the alternating series from before, which we'll now change by a sign:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

To each positive term at an odd index n, e.g. $\frac{1}{3}$ at $n = 3$, we can associate the negative term at $2n$, in this case $-\frac{1}{6}$; if we pair the terms like this, we get

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots$$

which simplifies to

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots ,$$

which is just half of the original sum

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right),$$

and so adds up to $\frac{1}{2}\log 2$. But we only rearranged terms!

In fact, more is true: by rearranging terms in a conditionally convergent series, it is possible to get *any* number as the eventual sum. Thus it is very important not to do anything that might change the ordering of more than finitely many terms!

For absolutely convergent series, though, rearrangement will do the expected thing, i.e. not change the sum. This is because the sort of maneuver above becomes impossible: it relies on the fact that choosing the correct signs will allow us to move the value of the series arbitrarily, but for absolutely convergent series the series still converges without any signs at all and so moving which go where does not affect the result.

Next time we'll talk about induction and get in a few more convergence tests, and maybe even say something about power series.