Lecture 14: sequences and series

Calculus II, section 3
April 4, 2022

Our final unit of the class is on sequences and series. For now, this is separate from our previous topics like derivatives, integrals, differential equations, arc length, etc., though at the end we'll tie some of them together through Taylor series.

First, what is a sequence? A formal definition is: a sequence is a function from the natural numbers (1, 2, 3, etc., or sometimes starting at 0 depending what's most convenient) to the real numbers (you could also use complex numbers if you want). What this means is that to every natural number n we assign a real number a_n.

More concretely, a sequence is a list of numbers $a_1, a_2, a_3, \ldots$. For example, a very boring sequence is $0, 0, 0, \ldots$. A slightly more interesting one is something like $1, 2, 3, 4, \ldots$, or $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$. (Spot the pattern!)

More generally yet, sequences don't need to have straightforward formulas. For example, you could define the sequence a_n to be the number of possible positions a chess board could be in after n moves, or the nth prime number, or the nth digit of π in base 31 $(3, 4, 12, 2, 5, 24, 14, 18, 5 \ldots)$. Another famous sequence is the Fibonacci sequence: $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$, so the sequence goes $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

For many of these sequences, it doesn't make sense to talk about their limits. For some it does, though: for an easy example, the limit of $0, 0, 0, \ldots$ is just 0. More generally, just like for functions we think of $\lim_{n\to\infty} a_n$ as the number that a_n approaches as $n \to \infty$. For example, $\lim_{n\to\infty} \frac{1}{n} = 0$.

In the case where $a_n = f(n)$ for some real function $f(x)$ and $\lim_{x\to\infty} f(x)$ exists, it is the same thing as $\lim_{n\to\infty} f(n)$. Sometimes this is easier to compute since we have access to things like L'Hopital's rule; sometimes it's equally hard either way.

There is a more formal definition of the limit, which you may have seen a version of for limits of functions. It is this: we say that a number L is the limit of a sequence a_n if for every $\epsilon > 0$, there exists some number N such that $|a_n - L|$ is always less than ϵ for $n > N$. This is really just a way of formalizing the idea that a_n gets arbitrarily close to L; don't worry about it too much if it doesn't make sense.

Let's look at a few more examples. Suppose that $a_n = \frac{n^2}{2n(n+1)}$, so starting from $n = 1$ the sequence looks like $\frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{5}{12}, \frac{3}{7}, \frac{7}{16}, \ldots$. If you write out the decimal approximations to these fractions, $0.25, 0.333, 0.375, 0.4, 0.417, 0.429, 0.438$, you might guess that this is converging to something in the neighborhood of 0.5, and indeed this is the case: one way to see this is to write $n(n+1) = n^2 + n$ and so $n^2 = n(n+1) - n$, so that

$$a_n = \frac{n(n+1) - n}{2n(n+1)} = \frac{1}{2} - \frac{1}{2(n+1)},$$

which tends to $\frac{1}{2}$ as $n \to \infty$.

Another example: consider the sequence $1, -1, 1, -1, \ldots$, which we can write as $a_n = (-1)^n$ starting at $n = 0$. Does this sequence converge?

A more complicated example is e.g. the sequence with $a_0 = 0$ and the rule $a_n = \frac{a_{n-1}+1}{2}$. You can work out that this gives $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$, which suggests that it might converge to 1. Indeed, you can check this: the recurrence relation fully determines the values, so if you can find any formula that satisfies the relation it must be true; and you can check that $1 - \frac{1}{2^n}$ works, so $a_n = 1 - \frac{1}{2^n}$ and therefore $\lim_{n\to\infty} a_n = 1$. Another way to see this is that if the sequence converges to some number L, then for n very large we must have $a_n \approx L$ and $a_{n-1} \approx L$, so $L \approx \frac{L+1}{2}$ with the approximation becoming better and better as $n \to \infty$. Since this doesn't depend on n, it must be an equality which we can solve to get $L = 1$.

There's a principle which is secretly at play in a lot of these sorts of problems, which is this. If we have a sequence a_n which is *monotonic*, meaning it is either constantly increasing or decreasing (or constant), and it is bounded, i.e. there are constants on either side which it must always be between, then it has to converge. For example, $a_n = e^n$ or $a_n = 1$ are both monotonic (increasing and constant respectively), but $a_n = 1$ is bounded while $a_n = e^n$ is not; meanwhile $a_n = \sin n$ is not monotonic but is bounded, and $a_n = \frac{n}{n+1}$ is both monotonic (increasing) and bounded, so it converges (to 1). There are various other rules paralleling the rules for limits of functions; for example, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ if both limits exist.

Okay, so we understand sequences, which may or may not have limits, which may be more or less complicated to compute. We now want to understand a related concept: infinite series.

To introduce this concept, consider the following problem from ancient Greek philosophy, Zeno's paradox. There are a number of paradoxes by this name, but the most straightforward one is this:[1] suppose I want to walk across this room. The first half is easy, but after walking the first half I then need to walk half of the remaining distance, and then half of what remains after that, and so on; because I have to do infinitely many segments, I'll never get there.

What's wrong with this? Well, let's write it as a math problem. Let's say that the length of the room is 1, so step 1 is to go a distance of $\frac{1}{2}$. After our next step, we've gone $\frac{1}{2} + \frac{1}{4}$; then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$; and so on. Thus what we're really doing is defining a sequence, with first term $\frac{1}{2}$, second term $\frac{1}{2} + \frac{1}{4}$, third term $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, and so on. Zeno's paradox is that even though as we keep going we add more and more terms, this sequence has a finite limit: after infinitely many steps, I've finished crossing the room, i.e. gone a distance of 1, and so the limit is 1.

This sort of situation in a limit is sufficiently common that we give it its own name: this is an infinite series, with terms $\frac{1}{2^n}$. More generally, for any sequence instead of just looking at it as a sequence we can take the *partial sums*: this gives a new sequence whose first term is a_1, then $a_1 + a_2$, then $a_1 + a_2 + a_3$, and so on. The limit of this new sequence is then $a_1 + a_2 + a_3 + \cdots$, which despite Zeno's intuition may be a finite number.

We use the following notation for this: the sum of the terms from $n = 1$ to $n = N$ for some number N is written as

$$\sum_{n=1}^{N} a_n.$$

[1] I'm changing the framing slightly to be less annoying to phrase in mathematical language, but it's the same idea.

For the limit, we write it as

$$\sum_{n=1}^{\infty} a_n.$$

Thus our example from Zeno's paradox with $a_n = \frac{1}{2^n}$ is just the computation

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

There are various other ways to see this result; one is to draw a square, cut off half, cut off half of what remains, and so on.

It's often also interesting to look at partial sums themselves, which we'll talk about more later, but primarily we'll be focused on infinite series

$$\sum_{n=1}^{\infty} a_n.$$

(The bounds could also be different; for example, we could start at $n = 0$, or $n = 2$, or anywhere.) The first question with any such series is whether or not it converges, i.e. whether the limit of the partial sums exists; the next question is what the limit is, if it exists.

Just like for integrals, there are a couple of ways for a series to diverge. Since series are over discrete bounds, the case where the integrand blows up at one of the endpoints is not an issue for series, but the other two are: a series diverges if either a_n fails to exist for some n (e.g. $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges without doing any complicated calculations, because the $n = 0$ term doesn't exist) or it gets small too slowly or not at all (e.g. $\sum_{n=1}^{\infty} 1$ diverges, because it is just $1 + 1 + 1 + \cdots = \infty$).

Let's generalize the example above where $a_n = \frac{1}{2^n}$: let $a_n = r^n$ for any real number r (so the previous case is $r = \frac{1}{2}$). When does

$$\sum_{n=1}^{\infty} r^n$$

converge, and when it does what does it converge to?

There's a few ways to see this, but the clearest is to look at the partial sums. Let

$$s_N = \sum_{n=1}^{N} r^n = r + r^2 + \cdots + r^N.$$

Observe that

$$rs_N = r^2 + r^3 + \cdots + r^{N+1} = s_{N+1} - r.$$

On the other hand, by definition $s_{N+1} = s_N + r^{N+1}$, so

$$rs_N = s_N + r^{N+1} - r$$

which we can solve to get

$$s_N = \frac{r^{N+1} - r}{r - 1}$$

whenever $r \neq 1$. Taking the limit as $N \to \infty$, we see that this converges if and only if $\lim_{N\to\infty} r^{N+1}$ converges, which is true if $|r| < 1$, false if $|r| > 1$, true at $r = 1$ and false at $r = -1$. We're excluding the case $r = 1$ anyway (for which we just have $s_N = N$), so this converges if and only if $|r| < 1$. In this case, $\lim_{N\to\infty} r^{N+1} = 0$ and so the limit tends to

$$\frac{-r}{r-1} = \frac{r}{1-r}.$$

In the special case where $r = \frac{1}{2}$, we get $\frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$ as above.

We started at $n = 1$ in this case to go with the above, but it's a little more natural to start at $n = 0$ so that the sum is $1 + r + r^2 + r^3 + \cdots$. We could go through the same calculation as above with this starting point, or just observe that this is adding 1 to the above to get

$$\sum_{n=0}^{\infty} r^n = \frac{r}{1-r} + 1 = \frac{1}{1-r}$$

whenever $|r| < 1$. This is called the geometric series.

One easy way of knowing that the geometric series can't converge when $|r| \geq 1$ is using the following principle: in order for $\sum_{n=1}^{\infty} a_n$ to converge, the limit of the a_n must exist and be equal to 0. This is because if we keep adding things for infinitely many terms, if there's any hope of the series converging they need to get increasingly small as we go or the sum will never stabilize.

However, this turns out not to be good enough: there are series with $\lim_{n\to\infty} a_n = 0$ such that $\sum_{n=1}^{\infty} a_n$ still does not converge. For example, consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We've avoided the bad spot $n = 0$; does this now converge?

It does not. This can be seen by grouping terms: the first two terms are $1 + \frac{1}{2}$, the next two terms are $\frac{1}{3} + \frac{1}{4}$, which are both at least $\frac{1}{4}$ and so the sum is at least $1 + \frac{1}{2} + 2 \cdot \frac{1}{4}$. The next four terms are all at least $\frac{1}{8}$ and so the sum is at least $1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8}$, and we can keep going in this fashion to keep adding terms which are at least $\frac{1}{2}$, so the series cannot converge.

We'll talk more next time about how to tell when series converge or diverge, and how to compute some partial sums. However there are some general properties to be aware of. For example, series are linear:

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n,$$

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

if both converge, and modifying/adding/removing finitely many terms does not affect whether or not a series converges.