# Homework 8

Calculus II, section 3

Optional; turn in by 11:59 PM Sunday March 27 to receive comments

Note: there was a mistake which probably made your lives harder, sorry about that: namely on 2 (d) the right-hand side should have been $e^{-x^2/2}$, not $e^{-x^2}$, since as is although it's possible to write the solution in terms of a (double) integral that integral can't be computed in terms of standard functions.

**Problem 1.** Solve the first-order equation $(x \log x)y' + y = x^n$, $y(1) = 1$ for any positive integer $n$.

**Solution.** First, divide by $x \log x$ to get the equation in standard form

$$y' + \frac{1}{x \log x} y = \frac{x^{n-1}}{\log x}.$$

We then multiply by the integrating factor

$$M(x) = e^{\int \frac{1}{x \log x}\, dx} = e^{\log \log x} = \log x$$

to get

$$y' \log x + \frac{1}{x} y = x^{n-1}.$$

Here we compute the integral $\int \frac{1}{x \log x}\, dx$ by substitution: if $u = \log x$, then $du = \frac{1}{x}\, dx$ and so the integral is $\frac{1}{u}\, du = \log u = \log \log x$. (We don't need a $+C$ because we just need one integrating factor, not every possible such function.)

The left-hand side should be $\frac{d}{dx}(M(x)y) = \frac{d}{dx}(y \log x)$, as can be confirmed by the product rule. Therefore we have

$$y \log x = \int x^{n-1}\, dx = \frac{x^n}{n} + C,$$

and so

$$y = \frac{x^n}{n \log x} + \frac{C}{\log x}.$$

We want to have $y(1) = 1$. However, if we plug in 1 both terms diverge!

To solve this problem, we want the singularities to cancel each other out, i.e. we combine terms to $\frac{x^n + Cn}{n \log x}$ and try to choose $c$ such that the limit as $x \to 1$ exists. Since $\log 1 = 0$, for this to be true the numerator needs to be 0, so $1 + Cn = 0$ and so $C = -\frac{1}{n}$ to give $y = \frac{x^n - 1}{n \log x}$.

Does this work? If we take the limit as $x \to 1$, by L'Hopital we get

$$\frac{nx^{n-1}}{n/x} = x^n \to 1,$$

and so not only does the limit exist but it is also equal to the desired value of $y(1)$. Therefore the answer is $y = \frac{x^n - 1}{n \log x}$, extended to the point $x = 1$, where it does not a priori exist, by continuity.

**Problem 2.** Consider a second-order equation like $y'' + F(x)y' + G(x)y = H(x)$. We hope to use the method of integrating factors to solve this equation; this will only work in very special cases.

(a) We want to find an integrating factor $M(x)$ such that if we multiply both sides the left-hand side $M(x)(y'' + F(x)y' + G(x)y)$ becomes equal to the second derivative $\frac{d^2}{dx^2}(M(x)y)$. Show that this is the same thing as requiring that $M(x)$ satisfy the two differential equations $2M'(x) = M(x)F(x)$ and $M''(x) = M(x)G(x)$.

(b) Find a (nontrivial) solution to the first differential equation $2M'(x) = M(x)F(x)$, potentially in terms of an integral (which you need not compute). (A trivial solution is given by $M(x) = 0$, but this won't be useful.)

(c) Using your solution to (b) and the equations from (a), show that we must have $4G(x) = F(x)^2 + 2F'(x)$ in order for this method to work.

(d) Consider the example $y'' + 2xy' + (x^2 + 1)y = e^{-x^2}$. Check that this satisfies the condition from (c) and use the method of integrating factors to find a solution with $y(0) = 1$ and $y(1) = 0$.

**Solution.**

(a) The second derivative $\frac{d^2}{dx^2}(M(x)y)$ by applying the product rule twice is $\frac{d}{dx}(M'(x)y + M(x)y') = M''(x)y + 2M'(x)y' + M(x)y''$. This is equal to $M(x)(y'' + F(x)y' + G(x)y) = M(x)y'' + M(x)F(x)y' + M(x)G(x)y$ if and only if all of the coefficients of $y$, $y'$, and $y''$ are the same, i.e. $M''(x) = G(x)$, $2M'(x) = M(x)F(x)$, and $M(x) = M(x)$, the last of which is obviously always satisfied, so this condition is the same as the two differential equations specified.

(b) Writing $M'(x) = \frac{dM}{dx}$, the differential equation is $2\frac{dM}{dx} = M(x)F(x)$, and multiplying by $dx$ and dividing by $M$ gives $\frac{2\,dM}{M} = F(x)\,dx$. Integrating gives $2\log M = \int F(x)\,dx$, and so $M(x) = e^{\frac{1}{2}\int F(x)\,dx}$.

(c) From the first differential equation, we know that $M'(x) = \frac{1}{2}M(x)F(x)$, and so $M''(x) = \frac{1}{2}(M'(x)F(x) + M(x)F'(x))$. Substituting the formula for $M'(x)$ again gives $M''(x) = \frac{1}{4}M(x)F(x)^2 + \frac{1}{2}M(x)F'(x)$. The second differential equation which $M$, $F$, and $G$ must satisfy is $M''(x) = M(x)G(x)$, so this is

$$\frac{1}{4}M(x)F(x)^2 + \frac{1}{2}M(x)F'(x) = M(x)G(x).$$

So long as $M(x)$ is not everywhere 0 (which it is not from the formula from (b)) we can divide by it to get $\frac{1}{4}F(x)^2 + \frac{1}{2}F'(x) = G(x)$; multiplying by 4 gives the desired equation.

(d) In this case, we have $F(x) = 2x$ and $G(x) = x^2 + 1$. Since $F'(x) = 2$, the equation from (c) says that

$$4G(x) = 4x^2 + 4 = (2x)^2 + 2 \cdot 2,$$

which is certainly true. Therefore the method of integrating factors should work, with

$$M(x) = e^{\frac{1}{2}\int 2x\,dx} = e^{x^2/2}.$$

Multiplying through by $e^{x^2/2}$, we can check that the left-hand side is indeed equal to $\frac{d^2}{dx^2}(e^{x^2/2}y)$, so

$$e^{x^2/2}y = \int \left( \int e^{-x^2/2}\,dx \right) dx.$$

Unfortunately, this is not possible to integrate in terms of elementary functions, because it is a mistake: the right-hand side of the equation was intended to read $e^{-x^2/2}$, not $e^{-x^2}$, so that when we multiply through by $e^{x^2/2}$ we are left with 1 on the right-hand side. As written, the best we could do is say that $y$ is $e^{-x^2/2}$ times this double integral.

Let's imagine, though, that the problem was written correctly, i.e. the differential equation was $y'' + 2xy' + (x^2+1)y = e^{-x^2/2}$. Then when we multiply by $M(x) = e^{x^2/2}$ we get $\frac{d^2}{dx^2}(e^{x^2/2}y) = 1$, and so

$$e^{x^2/2}y = \int\int 1\,dx\,dx = \int (x + C_1)\,dx = \frac{x^2}{2} + C_1x + C_2$$

for some constants $C_1$, $C_2$. Therefore

$$y = \frac{1}{2}x^2e^{-x^2/2} + C_1xe^{-x^2/2} + C_2e^{-x^2/2}.$$

Since we want to have $y(0) = 1$ and $y(1) = 0$, substituting $x = 0$ gives $C_2 = 1$, and $x = 1$ gives $0 = \frac{1}{2}e^{-1/2} + C_1e^{-1/2} + C_2e^{-1/2} = \left(C_1 + \frac{3}{2}\right)\frac{1}{\sqrt{e}}$ and so $C_1 = -\frac{3}{2}$, and so the solution is

$$y = \frac{1}{2}x^2e^{-x^2/2} - \frac{3}{2}xe^{-x^2/2} + e^{-x^2/2}.$$

**Problem 3.**

(a) Compute the Laplace transform of $f(x) = \sin x$.

(b) Use your answer to (a) and the Laplace transform to solve the differential equation $y''' + y = 1$, with $y(0) = 1$ and $y'(0) = y''(0) = 0$.

**Solution.**

(a) This is the integral $\int_0^\infty e^{-xs} \sin x \, dx$. We use integration by parts: if $u = e^{-xs}$ and $dv = \sin x \, dx$, then $du = -se^{-xs} \, dx$ and $v = -\cos x$, so the indefinite integral is

$$-e^{-xs} \cos x - s \int e^{-xs} \cos x \, dx.$$

Repeating the integration by parts, again with $u = e^{-xs}$ and now $dv = \cos x \, dx$, so $v = \sin x$, we get

$$\begin{aligned} &- e^{-xs} \cos x - s \left( e^{-xs} \sin x + s \int e^{-xs} \sin x \, dx \right) \\ = &- e^{-xs} \cos x - se^{-xs} \sin x - s^2 \int e^{-xs} \sin x \, dx. \end{aligned}$$

Evaluating at 0 and $\infty$, this is

$$1 - s^2 \int_0^\infty e^{-xs} \sin x \, dx.$$

The integral is the same as the original integral; if we call it $L$, we have an equation

$$L = 1 - s^2 L$$

and so

$$L(1 + s^2) = 1$$

and thus

$$L = \int_0^\infty e^{-xs} \sin x \, dx = \frac{1}{1 + s^2}.$$

(b) First, we want to apply the Laplace transform to each part. We know that $\mathcal{L}[y'] = -y(0) + s\mathcal{L}[y]$. Applying the same formula to $y'''$ gives

$$\begin{aligned} \mathcal{L}[y'''] &= -y''(0) + s\mathcal{L}[y''] \\ &= -y''(0) + s(-y'(0) + s\mathcal{L}[y']) \\ &= -y''(0) + s(-y'(0) + s(-y(0) + s\mathcal{L}[y])) \\ &= -y''(0) - sy'(0) - s^2 y(0) + s^3 \mathcal{L}[y]. \end{aligned}$$

In this case $y(0) = 1$ and $y'(0) = y''(0) = 0$, so

$$\mathcal{L}[y'''] = -s^2 + s^3 \mathcal{L}[y].$$

Therefore in all the left-hand side is

$$-s^2 + s^3 \mathcal{L}[y] + \mathcal{L}[y].$$

Meanwhile the right-hand side is $\mathcal{L}[1] = \frac{1}{s}$, so we have

$$-s^2 + s^3\mathcal{L}[y] + \mathcal{L}[y] = \frac{1}{s},$$

and thus

$$s^3\mathcal{L}[y] + \mathcal{L}[y] = (s^3 + 1)\mathcal{L}[y] = \frac{1}{s} + s^2 = \frac{s^3 + 1}{s}.$$

Dividing by $s^3 + 1$ gives

$$\mathcal{L}[y] = \frac{1}{s},$$

and so $y$ is the function whose Laplace transform is $\frac{1}{s}$, namely just $y = 1$. You can verify that this satisfies the differential equation and initial conditions.