Computing $\left(\frac{1}{2}\right)!$

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In class today, we found the formula

$$n! = \int_0^\infty x^n e^{-x}\,dx,$$

which is valid even for non-integer n, and I claimed that in fact this gives $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$. Let's actually prove this.

In fact, it turns out to be easier to compute a slightly different value: we use the formula $n! = n\cdot(n-1)!$ to find $(\frac{1}{2})! = \frac{1}{2}\cdot(-\frac{1}{2})!$ (the above formula is still valid even in this case!). Therefore it suffices to show that

$$\left(-\frac{1}{2}\right)! = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx$$

is equal to $\sqrt{\pi}$.

First, let's make a substitution: set $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then this integral is

$$\int_0^\infty \frac{1}{u} e^{-u^2}\cdot 2u\,du = 2\int_0^\infty e^{-u^2}\,du.$$

(Here since $u = \sqrt{x} = 0$ when $x = 0$ and $u = \infty$ when $x = \infty$ the bounds are the same.)

Now, notice that e^{-u^2} doesn't change upon replacing u by $-u$, since $e^{-(-u)^2} = e^{-u^2}$ since $(-u)^2 = u^2$. Therefore we can think of $2\int_0^\infty e^{-u^2}\,du$ as

$$\int_{-\infty}^0 e^{-u^2}\,du + \int_0^\infty e^{-u^2}\,du = \int_{-\infty}^\infty e^{-u^2}\,du.$$

This integral on the right is what we'll evaluate; let's call it X.

The trick here is this: instead of evaluating the integral X directly, we evaluate X^2. Let's use another variable for the second copy, say v; then we have

$$X^2 = \left(\int_{-\infty}^\infty e^{-u^2}\,du\right)\left(\int_{-\infty}^\infty e^{-v^2}\,dv\right).$$

As it turns out we can rearrange these integrals: $\int_{-\infty}^\infty e^{-v^2}\,dv$ is just a number, so we can put it inside the integral with respect to u to get

$$X^2 = \int_{-\infty}^\infty e^{-u^2}\left(\int_{-\infty}^\infty e^{-v^2}\,dv\right)du.$$

On the other hand, for each fixed u as far as the inner integral is concerned e^{-u^2} is just a number, and so we can move it inside that integral:

$$X^2 = \int_{-\infty}^\infty \left(\int_{-\infty}^\infty e^{-u^2} e^{-v^2}\,dv\right)du.$$

If we think of each integral as integrating over the whole real line, then this pair of integrals is saying for every real number u, we attach a real line with coordinate v and integrate over that. In other words we are integrating over all pairs of real numbers (u, v), i.e. integrating over the *plane*. (This is secretly multi-variable calculus, but don't tell.)

By the properties of the exponential function, the function we're integrating at the point (u, v) is $e^{-u^2}e^{-v^2} = e^{-(u^2+v^2)}$. Now, the distance of a point (u, v) in the plane from the origin $(0, 0)$ is given by the formula $\sqrt{u^2 + v^2}$ (by the Pythagorean theorem), so if we call this distance r then this function is just e^{-r^2}. What we want to do is work in polar coordinates, which we'll talk about more later into the semester; for now let's think of this as a new way of dividing up the plane into shells, just like we did to compute area and volume last time. We can think of the plane as what we get when we glue together circles of radius r for every r; if we do this only up to some fixed radius R, we get the area of a disk,

$$A = \int_0^R 2\pi r\, dr = 2\pi \cdot \frac{R^2}{2} = \pi R^2.$$

What we want to do is a bit different: we want to integrate our function over every point in the plane, but we've noticed that it actually only depends on r, and has value e^{-r^2}. Therefore the total contribution from all the points of radius r is $2\pi r e^{-r^2}$.

If we assemble all these together for all possible r, what we get is

$$X^2 = \int_0^\infty 2\pi r e^{-r^2}\, dr.$$

Now we can make another u-substitution: set $u = r^2$, so that $du = 2r\, dr$. Then this becomes

$$X^2 = \pi \int_0^\infty e^{-u}\, du,$$

and we computed in class that this integral is 1 so that in all

$$X^2 = \pi.$$

Since e^{-u^2} is always positive, X can't be negative, so it must be equal to $\sqrt{\pi}$.

Putting everything together, we have

$$\left(\frac{1}{2}\right)! = \frac{1}{2}\left(-\frac{1}{2}\right)! = \frac{1}{2}X = \frac{\sqrt{\pi}}{2}$$

as desired.