Pre-class worksheet 9: integrals in physics

Calculus I, section 10
Due November 28, 2023 by 4:10 PM

As we've discussed before, the integral should be thought of as a sort of "cumulative" version of the initial function. For example, if we know that a point is moving along a line with speed $v(t)$, then its position $x(t)$ satisfies

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\,dt,$$

i.e. we can recover the total distance traveled between t_1 and t_2 by the integral of the velocity between them. For example, if $v(t) = v$ is constant, then

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v\,dt = v(t_2 - t_1),$$

not surprisingly. If $t_1 = 0$, $t_2 = 1$, and $v(t) = 1 - 2t$, so that at first the point is moving in the positive direction but then stops and begins moving in the negative direction, we find that

$$x(1) - x(0) = \int_0^1 (1 - 2t)\,dt = (t - t^2)\Big|_0^1 = 0 - 0 = 0,$$

so after 1 second the point has returned to its original position. If we know the initial position $x(0)$, then we could determine $x(1)$ (in this case equal to $x(0)$; more generally, we could find any $x(t)$, as

$$x(t) = x(0) + \int_0^t v(s)\,ds$$

(replacing t by s in the integral since t is already in use).

In physics, it is common to know the acceleration of an object (using Newton's second law, $F = ma$) and want to know its position. For example, if the force is gravity and our object has height y at time t, the force is constant: $F = -mg$, where m is the mass and g is a gravitational constant, about 32 feet per second squared. Therefore acceleration $a(t) = y''(t) = \frac{F}{m} = -\frac{mg}{m} = -g \approx -32$ feet per second squared, so

$$v(t) = v(0) + \int_0^t a(s)\,ds.$$

Problem 1. If $v(0) = v_0$ is the initial velocity, use the formula above for the acceleration to compute $v(t)$ in terms of t, g, and v_0.

Problem 2. If $y(0) = y_0$ is the initial height, use the fact that $v(t) = y'(t)$ and your answer to problem 1 together with the method above (note that our position variable is now y instead of x) to compute $y(t)$ in terms of t, g, v_0, and t_0. You should recover our formula from the very first lecture, in the case $v_0 = 48$ feet per second and $y_0 = 4$ feet.