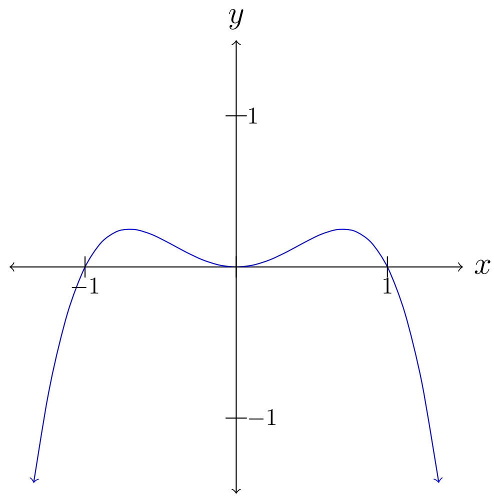
Pre-class worksheet 7: maximization

Calculus I, section 10
Due October 26, 2023 by 4:10 PM

Recall from the very first lecture we gave the following problem as an example of a question which is very hard if not impossible without calculus, but supposedly solvable in a reasonable way with calculus: what is the highest point on the graph of $y = f(x) = x^2 - x^4$?



In other words, for which x is $x^2 - x^4$ largest?

From looking at the graph, we might guess that there are actually two such points, which are negatives of each other; let's pick one and call it a. How can we find a?

One observation we can make is about the *derivative* of f near a. Since a is supposed to be a maximum (or at least a *local* maximum, i.e. $f(a)$ is greater than or equal to $f(x)$ for any x near a), if we pick x slightly greater than a then we must have $f(x) \leq f(a)$, so

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

On the other hand, if x is slightly less than a, we again have $f(x) \leq f(a)$ for the same reason, but now $x - a < 0$ and so

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0.$$

In order for f to be differentiable at a, these must be equal, so $f'(a) \leq 0$ and $f'(a) \geq 0$, i.e. $f'(a) = 0$.

This makes sense geometrically, too: for x a little less than a, since f has a maximum at a we expect f to be increasing at x, so $f'(x) > 0$; on the other hand for x a little more than a we expect f to now be decreasing, so $f'(x) < 0$. Since $f'(a)$ is in the middle, the maximum is the point where the derivative "crosses over," i.e. $f'(a) = 0$.

It's possible that this will occur at many points a. To find the maximum, we can simply compare the values $f(a)$ at all such points to see which is largest.

Problem 1. Find all points a with $f'(a) = 0$, where $f(x) = x^2 - x^4$ as above.

Problem 2. By checking the values of f at each solution a from the previous problem, determine for which real numbers $f(x) = x^2 - x^4$ is largest.