Lecture 4: limit laws and the squeeze theorem

Calculus I, section 10
September 14, 2023

Last time, we introduced limits and saw a formal definition, as well as the limit laws. Today we'll review limit laws from the worksheet and look at some one-sided limits, and introduce the squeeze theorem.

On the worksheet, we introduced the composition limit law: if $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x)) = \lim_{y\to L} f(y)$.

This lets us think about complicated limits piece-by-piece, which is very useful, but we have to be careful. For example, we might be tempted to say that we can use it to compute limits of the form

$$\lim_{x\to a} x \cdot f(x),$$

by first computing $\lim_{x\to a} f(x) = L$ (if it exists) and then $\lim_{x\to L} xL = L^2$. But this is generally not true: for example, if $f(x) = 1$, so $L = 1$, then

$$\lim_{x\to 0} xf(x) = \lim_{x\to 0} x = 0 \neq L^2 = 1.$$

What went wrong? The expression $xf(x)$ might look like a function composition, since we're feeding $f(x)$ into a machine to produce a new number. But actually this expression doesn't only depend on $f(x)$: it also depends on the original x! (This is now a multi-variable function, which might come up if you take through calculus 3, but definitely not in this class, and it won't have this nice behavior that composition of one-variable functions does here.) So the important thing to keep in mind when using the composition rule is to make sure that your expression is actually a composition of functions!

Even if the inside limit doesn't exist, we can still take advantage of this law if it goes to ∞ or $-\infty$. In this case we do treat ∞ as a number: if $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(g(x)) = \lim_{y\to\infty} f(y)$, and similarly for $-\infty$. For example, to compute

$$\lim_{x\to 0} 2^{-\frac{1}{x^2}},$$

we first compute

$$\lim_{x\to 0} -\frac{1}{x^2} = -\infty.$$

We then plug it in:

$$\lim_{x\to 0} 2^{-\frac{1}{x^2}} = \lim_{y\to -\infty} 2^y = 0.$$

You might complain at this point that last time I told you you don't have to worry about saying a limit goes to ∞ or $-\infty$, any such limit can just be said not to exist. That's true, and you could evaluate this limit without writing these symbols: just observe that as x goes to 0, $\frac{1}{x^2}$ gets larger and larger, so $2^{-\frac{1}{x^2}} = \frac{1}{2^{1/x^2}}$ gets smaller, since 2 to the power of a large number is big and the inverse of a big number is small. This notation with the ∞ symbols is just a way of keeping track of this sort of calculation, which you may find makes it easier for you; if it doesn't make it easier, feel free not to use it.

Next, let's come back to another concept we touched on last class: one-sided limits. We discussed before how if $\lim_{x\to a} f(x) = L$, this means that $f(x)$ approaches L as x goes to a from either direction, i.e. whether x is slightly less than a or slightly more than a we should have $f(x)$ close to L. We could instead weaken this requirement to only needing it to be true as x goes to a from one side or the other. We write

$$\lim_{x\to a^+} f(x)$$

for the limit as x goes to a from above, and

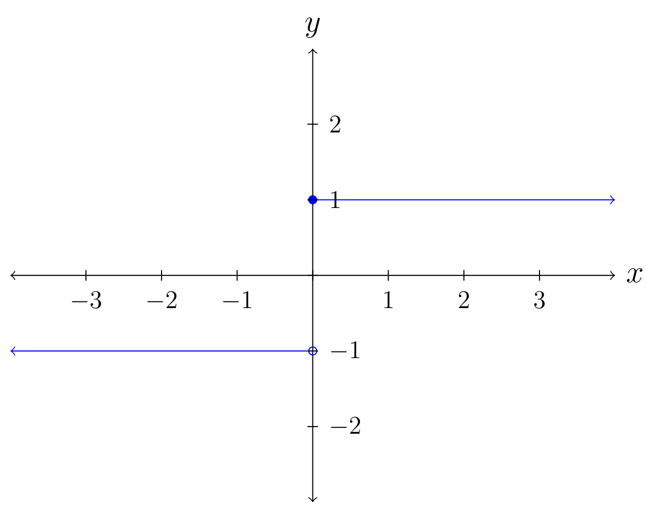
$$\lim_{x\to a^-} f(x)$$

for the limit as x goes to a from below. If $\lim_{x\to a} f(x)$ exists, then these one-sided limits must both exist and be the same; but it's possible that even if the total limit fails to exist, one or both of the one-sided limits may still exist (and if they both do, they may be different).

For example, consider the function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases},$$

whose graph looks like this.



As $x \to 0$ from above, the function is always 1, and so $\lim_{x\to 0^+} f(x) = 1$. But as $x \to 0$ from below, the function is -1, so $\lim_{x\to 0^-} f(x) = -1$.

Another common application of one-sided limits is to functions which do not exist on the whole domain and so can only be evaluated from one side. We saw an example last time involving logarithms; another example is

$$\lim_{x\to 0} \sqrt{x}.$$

Strictly speaking, even though we can plug in $x = 0$ to get $\sqrt{0} = 0$, this limit does not exist! This is because we can't approach it from below, only above, since $\sqrt{x}$ doesn't make sense for negative numbers.[1] If we replace the limit with a one-sided limit, $\lim_{x\to 0^+} \sqrt{x}$, then everything is as expected: this exists and is equal to 0.

A more complicated example is

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x+1}-1}.$$

Similarly, we need to require that the limit is only from above, since we can't plug in negative values to $\sqrt{x}$. Does this make the limit exist?

Well, the first thing to do is to get the square root out from the bottom, which we can do by conjugation:

$$\frac{\sqrt{x}}{\sqrt{x+1}-1} = \frac{\sqrt{x}}{\sqrt{x+1}-1} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{\sqrt{x}\sqrt{x+1}+\sqrt{x}}{x}.$$

Canceling a factor of $\sqrt{x}$, this is

$$\frac{\sqrt{x+1}+1}{\sqrt{x}},$$

and now as we take the limit as $x \to 0$ from above we see that this will blow up: the numerator goes to $\sqrt{1}+1 = 2$ while the denominator goes to 0.

Our final idea for the day is the squeeze theorem. This is based on the idea the limits respect inequalities: if $f(x) \le g(x) \le h(x)$, then (assuming all the limits exist)

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x).$$

In particular, suppose that we know that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then

$$L \le \lim_{x\to a} g(x) \le L$$

and so $\lim_{x\to a} g(x)$ must also be equal to L.

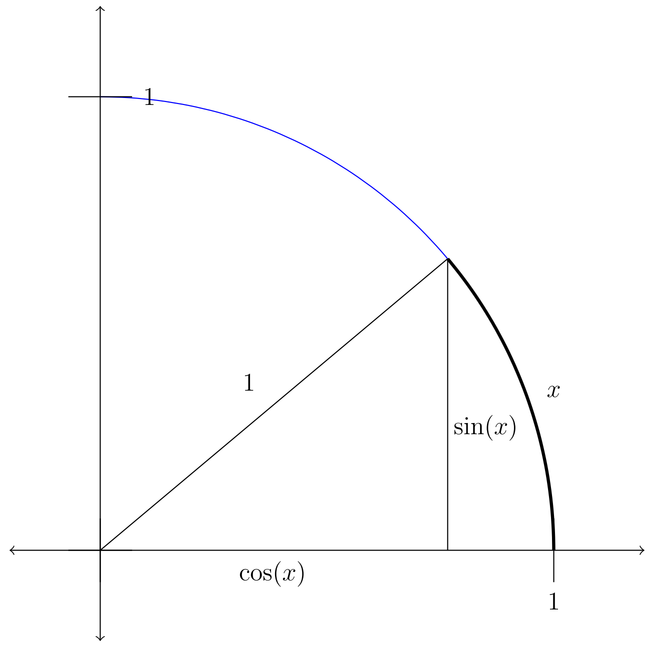
In fact, the squeeze theorem is a little stronger: we don't need to assume that the inner limit exists. If we have $f(x) \le g(x) \le h(x)$, at least for x sufficiently close to a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then it follows that $\lim_{x\to a} g(x) = L$. (The same thing works for one-sided limits.)

To see why such a thing might be useful, let's go back to the example I mentioned last class,

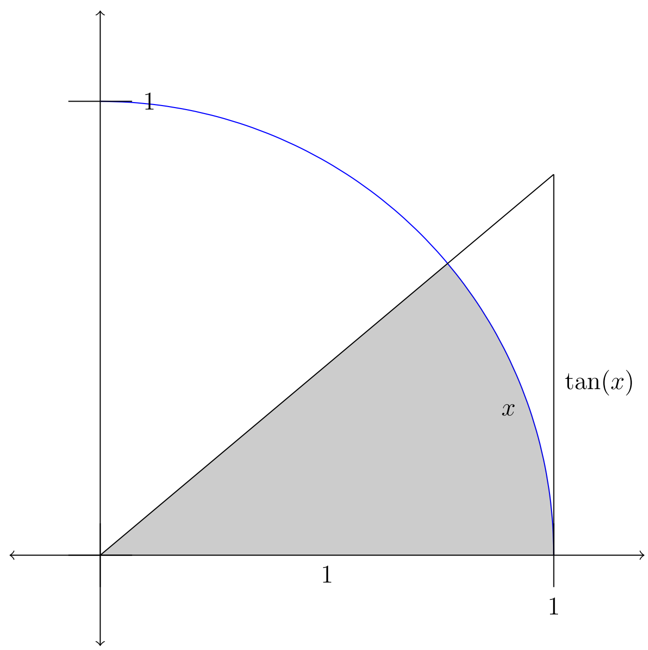
$$f(x) = \frac{\sin(x)}{x}.$$

I'm going to make two claims: at least for x small, $\frac{\sin(x)}{x} \ge \cos(x)$ and $\frac{\sin(x)}{x} \le 1$. Assuming everything is positive for simplicity, these are the same thing as $\tan x \ge x$ and $\sin(x) \le x$. To check that these are actually true, we can look at the unit circle:

[1] Again, if you allow complex numbers, you solve this problem at the price of introducing new ones.



The length of the vertical line ($\sin x$) must be less than the length of the curved line (x), so $\sin(x) \leq x$.



If we instead look at a larger triangle, the area of the whole triangle is $\frac{1}{2}\tan(x)$, while the area of the wedge is $\frac{x}{2\pi}$ of the area of the whole unit circle, which is π, and so the area of the wedge is $\frac{x}{2}$. Since the triangle contains the wedge, it follows that $\frac{x}{2} \leq \frac{1}{2}\tan(x)$, and so $\tan(x) \geq x$.

We could do the same thing for negative values (and take one-sided limits each way to see that they agree), or just add on absolute value signs.

Now that we know these bounds, so $\cos(x) \leq \frac{\sin x}{x} \leq 1$, we can apply the squeeze theorem: taking the limit as $x \to 0$, we have

$$\lim_{x \to 0} \cos(x) = \cos(0) = 1$$

and

$$\lim_{x \to 0} 1 = 1,$$

so without doing essentially any real limit work we get for free

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

This is, a priori, a very difficult statement!