# Lecture 17: Newton's method and first integrals

Calculus I, section 10

November 9, 2023

To finish our unit on applications of derivatives, we give a final application which again has the mysterious quality that it does not initially seem related to derivatives at all. Say we have some equation we want to solve, $f(x) = 0$; for a concrete example, take $f(x) = xe^x - 1 = 0$. Although it looks simple, this equation has no algebraic solution: we cannot write down a formula for the solution x using the operations we know! This is a frequent occurrence; sometimes even when we can write down an answer, it is simply too complicated to work with in practice. For example, $x^4 + 2x^3 - x^2 - 1 = 0$ has a unique positive real solution, which admits an exact formula involving roots which is nevertheless too complicated to write down on the blackboard (you can look it up if you so desire—it's even too complicated for me to copy here). However it is much easier to work with its numerical approximation $x \approx 0.8445$, which for many purposes is sufficient.

How can we numerically approximate solutions? The trick is as follows. Take the example above, $f(x) = xe^x - 1$. The graph looks like this:
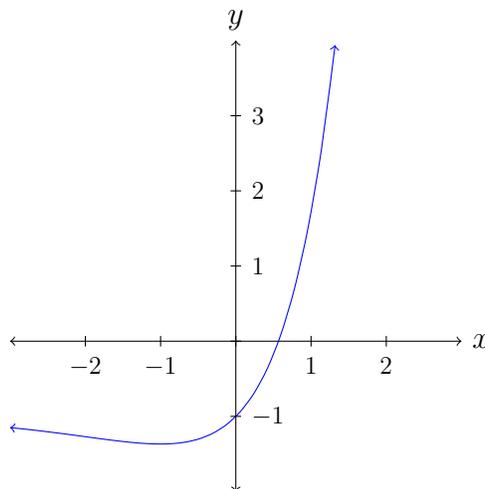
We're looking for the point where the curve crosses the x-axis.

Let's start by making a wild guess, which we call x_0; say $x_0 = 1$, which looks like it's probably not too far from the right answer. The idea is that by looking at the derivative, we can see in which direction we should change our guess in order to move towards the correct value. In particular, if we imagine that the true value x is close enough to x_0 that it is reasonable to linearly approximate f at x using the derivative at x_0, then we have $f(x) = 0 \approx f(x_0) + f'(x_0) \cdot (x - x_0)$. Of course, we don't know x, but if we imagine that this approximation is actually an equation then we could solve for it: we get $x \approx x_1 := x_0 - \frac{f(x_0)}{f'(x_0)}$. This is the point at which the line tangent to $y = f(x)$ at x_0 would intersect the x-axis.
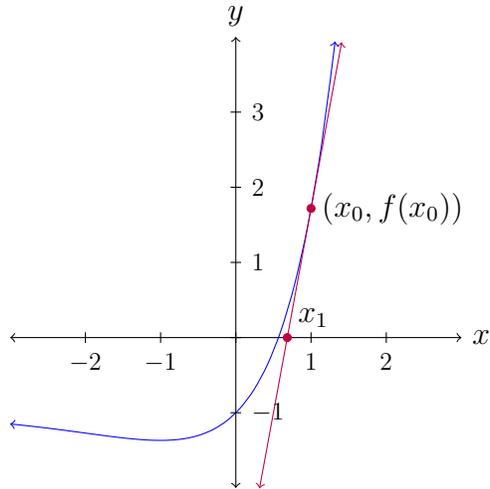
We can then repeat the process, with our new guess of x_1, to get $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, and so on. This is called Newton's method; it is the beginning of a whole field of numerical analysis.

In this case, we know $f(x) = xe^x - 1$ and $f'(x) = (x+1)e^x$, so we know everything we need to start computing. With $x_0 = 1$, we get

$$x_1 \approx 0.6839397,$$
$$x_2 \approx 0.5774545,$$
$$x_3 \approx 0.5672297,$$
$$x_4 \approx 0.5671433,$$
$$x_5 \approx 0.5671433,$$

and so on—after only a few steps we've converged to a value up to 7 decimal places, and one can check that plugging in these seven places gives $f(0.5671433) \approx 0.0000000265$, already very close to zero.

We could use the same technique on the other example mentioned: $f(x) = x^4 + 2x^3 - x^2 - 1 = 0$. Since we're looking for a positive solution, let's start by guessing $x_0 = 2$. We

have $f'(x) = 4x^3 + 6x^2 - 2x$ and so

$$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{27}{52} \approx 1.48077,$$

$$x_2 = 1.48077 - \frac{f(1.48077)}{f'(1.48077)} \approx 1.13098,$$

$$x_3 = 1.13098 - \frac{f(1.13098)}{f'(1.13098)} = 0.93005,$$

$$x_4 = 0.93005 - \frac{f(0.93005)}{f'(0.93005)} \approx 0.85488,$$

$$x_5 = 0.85488 - \frac{f(0.85488)}{f'(0.85488)} \approx 0.84468,$$

$$x_6 = 0.84468 - \frac{f(0.84468)}{f'(0.84468)} \approx 0.8445,$$

so we've gotten there after 6 iterations (and would need another to see that these digits aren't changing). Of course, we could keep going to get more and more accuracy.

Why did it take longer this time even though it seems like a simpler function? Because the point we started with was further away, so several iterations are spent just getting close enough to start getting accuracy. Even so, we get pretty good accuracy within a few iterations.

This is hinting at an important point: Newton's method is highly dependent on the initial choice of point. You can imagine that in our second example, if we'd chosen a different starting point we might have gotten a different solution of the equation. More concerningly, in our first example a poor choice of point would mean Newton's method would never converge!

Let's see what happens if we tried taking $x_0 = -1$ in our first example. We get $f(x_0) = -\frac{1}{e} - 1$ and $f'(x_0) = 0$, so Newton's method cannot even get started! This will always happen if we choose x_0 with $f'(x_0) = 0$, so that's a situation we want to make sure to avoid.

Okay, what about $x_0 = -2$? Then we have

$$x_1 \approx -11.3891,$$
$$x_2 \approx -8516.92,$$

and so forth—the x_n are rapidly diverging to $-\infty$! This is picking up on the fact that $f(x)$ does increase, as if moving towards zero, as $x \to -\infty$—but in fact not only does it never get towards zero, but it is not even asymptotic to it, as $\lim_{x \to -\infty} f(x) = -1$. (We could see this by L'Hôpital's rule: $xe^x - 1 = \frac{x - e^{-x}}{e^{-x}}$, whose limit is then the same as that of $\frac{1 - e^{-x}}{e^{-x}} = e^x - 1$, which tends to -1 as $x \to -\infty$.) In short: Newton's method is very powerful, but also very easy to fool; it only works well if you have a reasonably well-behaved function and make a good choice of initial guess.

One interesting diversion: what would happen if we apply Newton's method to $f(x) = x^2 + 1$, which has no real roots?

We don't want to start at $x_0 = 0$, since there $f'(0) = 0$, so let's start at $x_0 = 1$. Then we find

$$x_1 = 1 - \frac{1^2 + 1}{2 \cdot 1} = 0,$$

so we have to stop there—we know we can't plug in 0 to Newton's method. We might guess that this sort of thing always happens. But in fact reality is weirder: if we start with $x_0 = 2$, then we get

$$x_1 = 2 - \frac{2^2 + 1}{2 \cdot 2} = \frac{3}{4},$$
$$x_2 = \frac{3}{4} - \frac{(3/4)^2 + 1}{2 \cdot \frac{3}{4}} = -\frac{7}{2}4 \approx -0.29,$$
$$x_3 \approx 1.57,$$
$$x_4 \approx 0.47,$$
$$x_5 \approx -0.83,$$
$$x_6 \approx 0.19,$$
$$x_7 \approx -2.54,$$

and so on—bouncing around seemingly at random and never converging anywhere. This is an example of a *chaotic system*: its behavior is highly sensitive to the initial choice of starting point, and in this case will either blow up after a few iterations or oscillate apparently forever. In fact it is possible to derive an exact formula for these values via some trigonometric identities: if $x_0 = \cot \theta$, then $x_n = \cot(2^n \theta)$. Once n is reasonably large, this oscillates very rapidly in the initial choice θ, so it is not surprising that this gives a chaotic system.

We're now ready to start our final unit, where we focus on an apparently completely different problem, again defined by limits, which is in some ways the opposite problem to differentiation. When taking derivatives, we zoom in on a particular point and ask how our function is changing instantaneously at that point. Our new problem, integrals, asks us to zoom out and look at the cumulative total: what is the "continuous sum" of the values of our function so far?

The easiest way to think of integrals is as measuring area under a curve. Suppose we have some function $f(x)$.
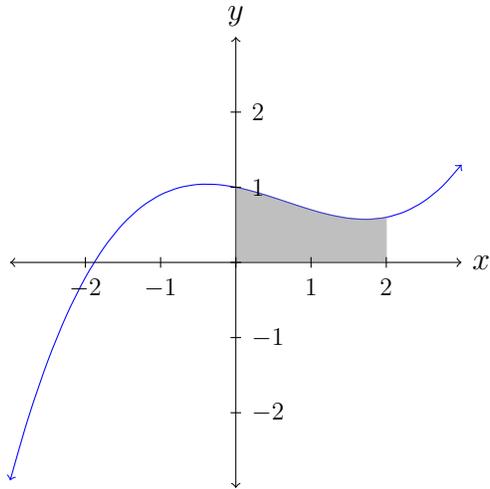
What is the *cumulative* area under the graph of the function from 0 to 2, i.e. the area of the shaded region?

For simple functions, this is easy to find. For example, if $f(x)$ is constant, we can evaluate the area using the formula for the area of a rectangle: if $f(x) = 1$,
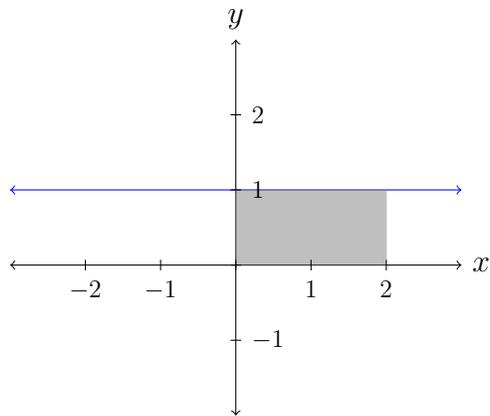

then the area is just $2 \cdot 1 = 2$. More generally, if $f(x) = c$ is constant, then the area between the graph of $f(x)$ and the x-axis between two points $x = a$ and $x = b$ is $c \cdot (b - a)$.
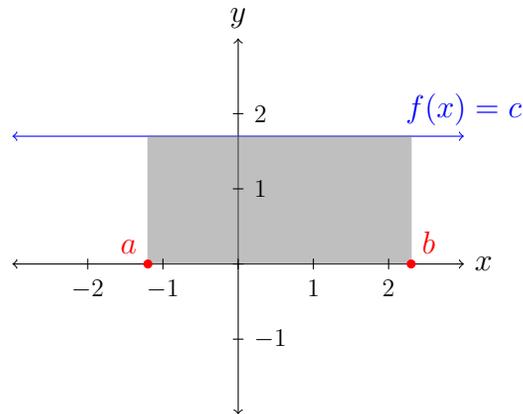
What about linear functions? In some cases this is easy geometry: if $f(x) = x$, then the area under $y = f(x)$ between $x = 0$ and $x = b$ is just the area of the right triangle with both side lengths b, i.e. $\frac{1}{2}b^2$.
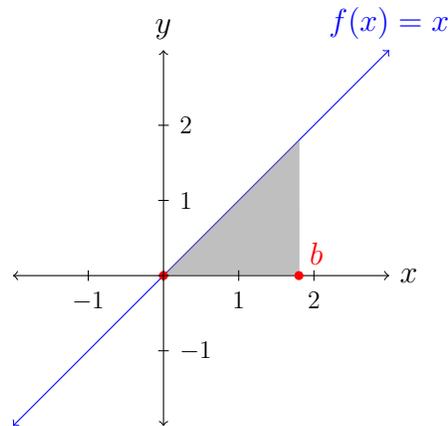


For general linear functions, we have to do a little more work, but not too bad: if $f(x) = mx + c$, then the area under $y = f(x)$ between $x = a$ and $x = b$ is the area of a trapezoid,
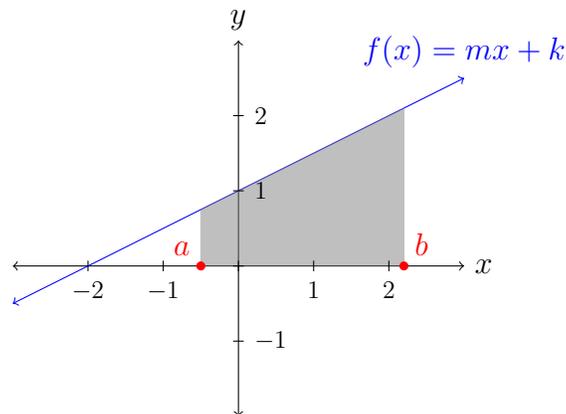

which you might recall from geometry has area $(b - a) \cdot \frac{f(a)+f(b)}{2} = (b - a) \cdot \frac{m(b+a)+2k}{2} = \frac{1}{2}m(b^2 - a^2) + k(b - a)$.

For more complicated functions, there's not much we can do. Even for the next-simplest polynomial $f(x) = x^2$, how could we evaluate just the area under $y = x^2$ between $x = 0$ and $x = 1$?
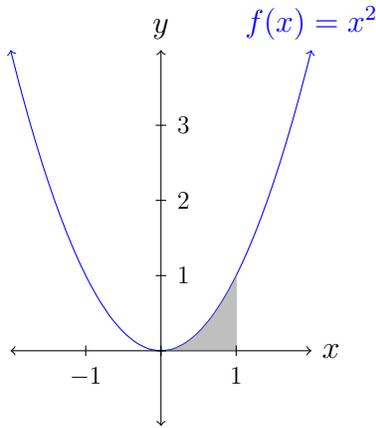
We could compute the areas under linear functions, but it was kind of messy; the really nice case was *constant* functions, so let's try approximating $f(x) = x^2$ by a constant function on this region. (This is the zeroth-order approximation.) There are a few different constant approximations we could choose; the most natural are the beginning of the interval, the end, or the middle, i.e. we approximate $f(x) = x^2$ between 0 and 1 by either $f(0) = 0$, $f(1) = 1$, or $f(\frac{1}{2}) = \frac{1}{4}$. On average, we might guess that the middle gives the best approximation, and this sort of consideration is important for numerical calculation, but for us it won't matter so we'll take the end of the interval because it's easiest to visualize. Thus our approximation is the length of the interval, $1 - 0 = 1$, times the height of the constant function, i.e. 1, which is just 1.
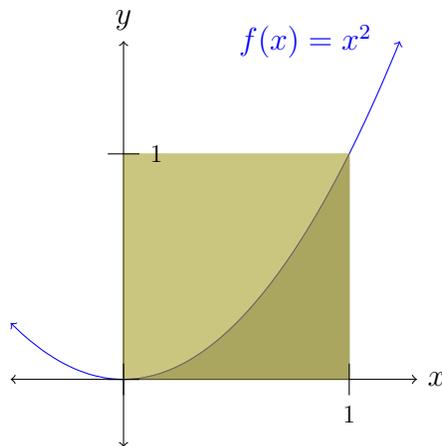

This doesn't seem like a great approximation; it looks like an overestimate. To do better, instead of a constant function we can use *two* constant functions: we approximate $f(x) = x^2$ between 0 and $\frac{1}{2}$ by $f(\frac{1}{2}) = \frac{1}{4}$, and between $\frac{1}{2}$ and 1 by $f(1) = 1$.
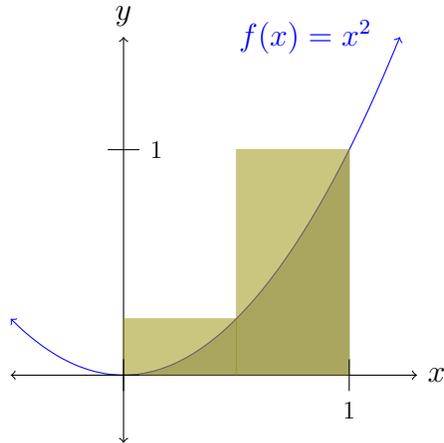
Now we approximate the true area by adding up the area of the rectangles: $\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 = \frac{5}{8} = 0.625$.

This still looks like an overestimate, so we can divide the range further, now into four rectangles:
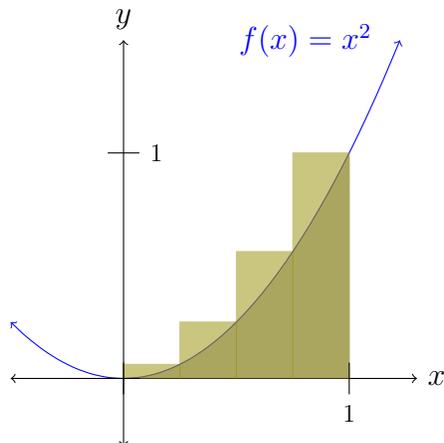
the new estimate for the area is $\frac{1}{4} \cdot \frac{1}{16} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{9}{16} + \frac{1}{4} \cdot 1 = \frac{15}{32} = 0.46875$.

This is still an overestimate, but it definitely seems like we're getting closer. We could also approach from the other direction: if we take our constants to be the values at the beginnings of the interval instead of the ends, after dividing the interval into four pieces here the picture would look as follows:
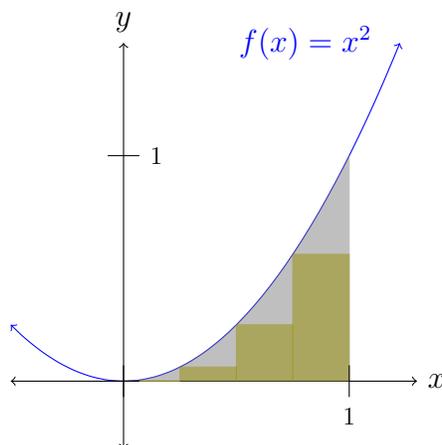
for an estimate of $\frac{1}{4} \cdot \frac{1}{16} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{9}{16} = \frac{7}{32} = 0.21875$. This looks like an underestimate from the graph, so the true value must lie somewhere in between $\frac{7}{32}$ and $\frac{15}{32}$. If we kept subdividing the interval, we could find much tighter bounds: for example if we take 100 rectangles, we'll find that the area must be between 0.32835 and 0.33835. This is suggestive that it might be converging to $0.333\ldots = \frac{1}{3}$, but it isn't converging so fast that this is easy to see by hand.

Before we look further into more powerful techniques to compute such things, let's think a little more about what we're doing. If we have any interval $[a, b]$ and a function $f(x)$, we want to compute the area under the curve $y = f(x)$ between $x = a$ and $x = b$. Let's give this a name:

$$\int_a^b f(x)\, dx,$$

the *(definite) integral* of $f(x)$ from a to b. (The dx is to show what variable we're integrating with respect to, like in $\frac{d}{dx}$.) We can estimate this area by dividing our interval into N sections, each of which then has width $w = \frac{b-a}{N}$, and we can take two approximations: the right Riemann sum

$$R_{a,b,N}^+(f) = w \cdot f(a + w) + w \cdot f(a + 2w) + \cdots + w \cdot f(a + (N-1)w) + w \cdot f(a + Nw)$$

where $f(a + Nw) = f(a + N \cdot \frac{b-a}{N}) = f(b)$, and the left Riemann sum

$$R_{a,b,N}^-(f) = w \cdot f(a) + w \cdot f(a + w) + w \cdot f(a + 2w) + \cdots + f(a + (N-1)w),$$

where $f(a + (N-1)w) = f(a + (N-1) \cdot \frac{b-a}{N}) = f(b - w)$. These correspond to our upper and lower bounds for $f(x) = x^2$ above. It won't always be true that $R_{a,b,N}^+(f) \geq R_{a,b,N}^-(f)$; if f is decreasing instead of increasing, the reverse will be true. Nevertheless, we expect that the "true" value $\int_a^b f(x)\, dx$ will be between the two Riemann sums, just as in the example above.

To get a better and better approximation, we take $N \to \infty$, or equivalently the width w of our rectangles to 0. If $\lim_{N \to \infty} R_{a,b,N}^+(f)$ and $\lim_{N \to \infty} R_{a,b,N}^-(f)$ both exist and are equal,

then we say that $f(x)$ is Riemann-integrable (or just integrable) on the interval $[a, b]$.[1] In this case we define $\int_a^b f(x)\,dx$ to be equal to this limit.

For example, in the case above we have $a = 0$, $b = 1$, and $f(x) = x^2$. Thus

$$R_{a,b,N}^+(f) = wf(w) + wf(2w) + \cdots + wf(Nw)$$

with $w = \frac{1}{N}$, so $wf(Nw) = wf(1)$; since $f(x) = x^2$, this is

$$w^3 + 4w^3 + 9w^3 + \cdots + N^2 w^3 = \frac{1}{N^3}\left(1 + 4 + 9 + \cdots + N^2\right).$$

On the other hand

$$\begin{aligned}
R_{a,b,N}^-(f) &= wf(0) + wf(w) + \cdots + wf((N-1)w) \\
&= 0 + w^3 + 4w^3 + \cdots + (N-1)^2 w^3 \\
&= \frac{1}{N^3}\left(1 + 4 + 9 + \cdots + (N-1)^2\right).
\end{aligned}$$

Therefore

$$R_{a,b,N}^+(f) - R_{a,b,N}^-(f) = \frac{N^2}{N^3} = \frac{1}{N} \to 0$$

as $N \to \infty$, so we expect the integral to exist.

Saying what it is is a little trickier. There is a neat trick, which you can try to figure out on your own if you want, that

$$1 + 4 + 9 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6},$$

which for N large has main term $\frac{2N^3}{6} = \frac{N^3}{3}$, so $\lim_{N\to\infty} R_{a,b,N}^+(f) = \frac{N^3/3}{N^3} = \frac{1}{3}$, as we guessed before. However this calculation relies on this trick about sums of squares, and for more complicated functions we won't have such tricks available in general. We'd like a more powerful method; we'll get one next time, called the fundamental theorem of calculus.

---

[1] Riemann integration, the kind we're doing here, isn't the only kind of integration; for example Lebesgue integration uses horizontal rectangles instead of vertical ones, and is able to converge to some integrals which wouldn't exist via the Riemann sum method. In this course, though, we'll only see Riemann integrals, and they'll suffice for all practical purposes.