Lecture 16: mean value theorem and L'Hôpital's rule

Calculus I, section 10

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On the worksheet for today, we encountered Rolle's theorem: if $f(a) = f(b) = 0$ and f is differentiable between a and b, then there must be a point c between a and b such that $f'(c) = 0$.

The proof doesn't really depend on $f(a)$ and $f(b)$ being 0, just on them being the same, so we could guess how to generalize the theorem. In fact we could generalize it even more:

Theorem (Mean Value Theorem)**.** *If f is differentiable between a and b (and continuous at a and b), then there exists some value c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

In the case where $f(a) = f(b) = 0$, this recovers Rolle's theorem: then $f'(c) = 0$. Indeed, the same thing works so long as $f(a) = f(b)$.

More generally, what this is saying is that we can, in some sense, combine the information of the derivatives of f at various points—all of which is purely *local* information—to be able to say something about the difference between values of f at different points, which is *global* information. For example, if we know that $f'(x) \geq 5$ for $a \leq x \leq b$, then it follows from the mean value theorem that $\frac{f(b)-f(a)}{b-a} = f'(c) \geq 5$, so $f(b) - f(a) \geq 5(b-a)$, i.e. $f(b) \geq f(a) + 5(b-a)$.

Why should this theorem be true? Well, it really boils down to Rolle's theorem: we want to "tilt" the graph of a given function so that the endpoints have the same value. In particular, if f is as in the statement of the mean value theorem, let $r = \frac{f(b)-f(a)}{b-a} = \frac{\Delta f}{\Delta x}$ be the overall slope, and define a new function

$$F(x) = f(x) - r(x-a) - f(a).$$

Then $F(a) = f(a) - 0 - f(a) = 0$ and $F(b) = f(b) - \frac{f(b)-f(a)}{b-a} \cdot (b-a) - f(a) = f(b) - (f(b) - f(a)) - f(a) = 0$, and F is differentiable with derivative

$$F'(x) = f'(x) - r,$$

so we can apply Rolle's theorem: there is some point c between a and b such that $F'(c) = 0$. Since $F'(c) = f'(c) - r = 0$, it follows that $f'(c) = r$, so the mean value theorem is proven.

Note that this is an example of what's called a non-constructive proof: even though we've shown that there must exist some c with this property between a and b, we have no idea what it is!

An example is $f(x) = \sin(x)$ between $a = 0$ and any point $x > 0$. We have $\frac{\sin(x)-\sin(0)}{x-0} = \frac{\sin(x)}{x}$, and by the mean value theorem this must be equal to $f'(c) = \cos(c)$ for some $0 \leq c \leq x$. Since $\cos(c) \leq 1$ for all c, we get the inequality $\sin(x) \leq x$ for free, without having to do any extra geometry.

Another application is something which might seem obvious, but is actually quite hard to prove otherwise, and which will be the foundation of integration theory later in the course:

if $f'(x) = 0$ between a and b, then $f(x)$ is constant between a and b. Indeed, for any x in the interval we can apply the mean value theorem with endpoints a and x: $\frac{f(x)-f(a)}{x-a} = f'(c) = 0$ for some c in the interval since $f'(c) = 0$ for all such c, so $f(x) = f(a)$ for every x in the interval.

The most common application of the mean value theorem, though, is L'Hôpital's rule. Before we see how to derive it from the mean value theorem, let's first say what it is.

When we talked about limits earlier in the semester, there were basically three kinds of situations: limits which straightforwardly approach a number, like $\lim_{x\to 1} \frac{x}{x+1} = \frac{1}{1+1} = \frac{1}{2}$; limits which go to infinity in one direction or the other, like $\lim_{x\to 0} \frac{1}{x}$ or $\lim_{x\to\infty} x + 1$; or limits which go to something ambiguous where we have to do more work, like $\lim_{x\to 1} \frac{x^2-1}{x+1}$ or $\lim_{x\to+\infty} xe^{-x}$, which if we just plug in the value often give something like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, or other expressions which can evaluate to various different things (including not existing). The idea of L'Hôpital's rule is to help us in this last situation.

The most straightforward example is when we're trying to evaluate something of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where f and g exist and are differentiable at a and $f(a) = g(a) = 0$. (If $g(a)$ is nonzero, then the limit is just a number $\frac{f(a)}{g(a)}$, and if $f(a)$ is nonzero while $g(a) = 0$ then the limit approaches something of the form $\frac{f(a)}{0}$ which does not exist, so this is the most interesting situation.) In this case, L'Hôpital's rule says that we can replace f and g with their derivatives:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

For example, consider

$$\lim_{x\to 0} \frac{e^x - 1}{x}.$$

We have $e^0 - 0 = 0$, so both the numerator and the denominator go to zero and are differentiable, so L'Hôpital's rule applies: we get that this limit is equal to

$$\lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1.$$

Sometimes, although the limit we start with is indeterminate, after applying L'Hôpital's rule we end up in a situation where either the limit can be evaluated straightforwardly (as above, which is the situation we want to be in) or straightforwardly does not exist. For example,

$$\lim_{x\to 0} \frac{\sqrt{x}}{\sin(x)}$$

is at first indeterminate: $\frac{\sqrt{0}}{\sin(0)} = \frac{0}{0}$. If we apply L'Hôpital's rule, we get

$$\lim_{x\to 0} \frac{\frac{1}{2\sqrt{x}}}{\cos(x)} = \lim_{x\to 0} \frac{1}{2\sqrt{x}\cos(x)},$$

and plugging in 0 gives $\frac{1}{2\sqrt{(0)}\cdot\cos(0)} = \frac{1}{0}$, so this limit does not exist.

Sometimes, neither situation will occur, and our limit is still ambiguous. In this case, we can simply apply L'Hôpital's rule again, and in fact in practice we often have to apply it multiple times. For example,

$$\lim_{x\to 1} \frac{\ln(x) + 1 - x}{(x-1)^2}$$

starts off ambiguous: $\frac{\ln(1)+1-1}{(1-1)^2} = \frac{0}{0}$. After applying L'Hôpital's rule, it is

$$\lim_{x\to 0} \frac{\frac{1}{x} - 1}{2(x-1)}.$$

If we plug in 1 to both numerator and denominator, we again get $\frac{0}{0}$: this is still ambiguous! It is now something we could evaluate by hand, but we can also apply L'Hôpital's rule again to get

$$\lim_{x\to 0} \frac{-\frac{1}{x^2}}{2} = -\frac{1}{2}.$$

We next want to prove L'Hôpital's rule, but let's first pause to observe how remarkable this is. We first introduced limits in order to approach the concept of a derivative; it now turns out that derivatives provide a very powerful tool to evaluate limits, and we could think of all of differential calculus as really being about evaluating these kinds of limits to better study functions, with derivatives just a tool along the way. In other words, derivatives and limits are equivalent in a sense: we use limits to define derivatives and derivatives to compute limits.

To prove L'Hôpital's rule, we use the mean value theorem. In fact we want a slightly stronger form of it. Instead of looking at $\frac{f(b)-f(a)}{x-b}$, we could ask in general about $\frac{f(b)-f(a)}{g(b)-g(a)}$. It turns out that this is equal to $\frac{f'(c)}{g'(c)}$ for some c between a and b; thus the case $g(x) = x$ recovers the usual mean value theorem.

To prove this generalization, consider $F(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$. Then $F'(x) = (f(b)-f(a))g'(x) - (g(b)-g(a))f'(x)$ and so by the (usual) mean value theorem for some c between a and b we must have

$$\begin{aligned}\frac{F(b)-F(a)}{b-a} &= \frac{(f(b)-f(a))g(b) - (g(b)-g(a))f(b) - (f(b)-f(a))g(a) + (g(b)-g(a))f(a)}{b-a} \\ &= 0 \\ &= (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c)\end{aligned}$$

so

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Now suppose that we're in the situation of L'Hôpital's rule, where $f(a) = g(a) = 0$. Then by the generalized mean value theorem for some c between a and b we have $\frac{f(b)}{g(b)} = \frac{f'(c)}{g'(c)}$. As

$b \to a$, since c is between a and b it must also go to c, so

$$\lim_{b\to a} \frac{f(b)}{g(b)} = \lim_{c\to a} \frac{f'(c)}{g'(c)}.$$

Renaming the limiting variables to x gives L'Hôpital's rule.

L'Hôpital's rule also works for $a = \pm\infty$, and when the indeterminacy is of the form $\frac{\infty}{\infty}$ instead of $\frac{0}{0}$; one way to see this is to take reciprocals of both sides, so that if $f(x)$ and $g(x)$ both tend to ∞ as $x \to a$, then

$$\frac{f(x)}{g(x)} = \frac{1/g(x)}{1/f(x)}$$

and so by L'Hôpital's rule the limit is the same as that of the ratio of derivatives

$$\frac{-g'(x)/g(x)^2}{-f'(x)/f(x)^2} = \frac{g'(x)}{f'(x)} \cdot \frac{f(x)^2}{g(x)^2}.$$

If the original limit goes to L, then so is this limit; but the second factor goes to L^2, so $\frac{g'(x)}{f'(x)}$ must go to $\frac{1}{L}$, i.e. $\frac{f'(x)}{g'(x)} \to L$, i.e. it agrees with the original limit.

For example, $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.

Note that L'Hôpital's rule will not always work. Consider

$$\lim_{x\to\infty} \frac{x + \sin(x)}{x + \cos(x)}.$$

We can evaluate this limit by hand: $\sin(x)$ and $\cos(x)$ are both bounded between 1 and -1, so the main term in the numerator and denominator is just x, so this goes to 1; more precisely, we could bound it by $\frac{x-1}{x+1} \leq \frac{x+\sin(x)}{x+\cos(x)} \leq \frac{x+1}{x-1}$ for all x, so by the squeeze theorem since both sides go to 1 so does the middle. On the other hand, we could try applying L'Hôpital's rule: it would tell us that the limit is the same as the limit of

$$\frac{1 + \cos(x)}{1 - \sin(x)},$$

or differentiating again

$$\frac{-\sin(x)}{-\cos(x)},$$

neither of which exists.

Finally, we can also apply our new generalized mean value theorem directly. In particular, recall our formula for first-order approximation:

$$f(x) \approx f(a) + f'(a) \cdot (x - a).$$

How good is this approximation?

Let's say $e(x) = f(x) - f(a) - f'(a)(x - a)$ is the error, so that

$$f(x) = f(a) + f'(a)(x - a) + e(x).$$

If the original formula is for first-order approximation, this is a kind of second-order correction, so we guess that $e(x)$ should be somehow related to $g(x) := (x - a)^2$. Indeed, we have $e(a) = 0$ and $e'(a) = f'(a) - f'(a) = 0$, same as for $g(x)$. If we apply the generalized mean value theorem to e and g, we get that for some c between a and x we have

$$\frac{e(x) - e(a)}{g(x) - g(a)} = \frac{e(x)}{g(x)} = \frac{e'(c)}{g'(c)}.$$

Since $e'(a) = g'(a) = 0$, we can apply the same theorem to see that there must be another constant C between a and c (and therefore between a and x) such that

$$\frac{e(x)}{g(x)} = \frac{e'(c)}{g'(c)} = \frac{e''(C)}{g''(C)} = \frac{f''(C)}{2},$$

so

$$e(x) = g(x) \cdot \frac{f''(C)}{2} = \frac{1}{2} f''(C)(x - a)^2.$$

Therefore

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(C)(x - a)^2$$

for some C between a and x.

If we can say something about f'' between a and x, then we can use that to say something about the error. For example, if $f(x) = e^x$, so $f''(x) = e^x$, then between -1 and 0 we have $f''(x) \leq f(0) = e^0 = 1$, and so the approximation $e^x \approx 1 + x$ for $-1 \leq x \leq 0$ has error $e(x) = \frac{1}{2} e^C x^2$ for some $-1 \leq C \leq 0$, and in particular $|e(x)| \leq \frac{1}{2} x^2$. Thus for x very close to 0 this is a good approximation; for x closer to -1, it can be significantly off. There is a more sophisticated such problem on your homework.

More generally, if we're interested in approximating $f(x)$ on some interval I and we know that on this region, $|f''(x)| \leq M$ for some fixed constant M, then

$$|e(x)| = \frac{1}{2} |f''(C)|(x - a)^2$$

as above must satisfy

$$|e(x)| \leq \frac{M}{2} \cdot (x - a)^2.$$