Homework 8

Calculus I, section 10

Due November 8, 2023 by 4:10 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

Problem 1. Evaluate the following limits using L'Hôpital's rule.

(a) $\lim_{x\to 0} \frac{\sin(x)-\tan(x)}{x^3}$

(b) $\lim_{x\to +\infty} \frac{x^3}{e^x}$.

The above problem is primarily directed towards Objective 9 (L'Hôpital's rule/mean value theorem).

Problem 2. Let $f(x) = x\sin(x)$.

(a) Show that for $0 \le x \le 0.01$, we have $|f''(x)| \le 2$.

(b) Using the formula for the error in first-order approximation together with part (a), conclude that $|f(x)| \le x^2$ for $0 \le x \le 0.01$.

The above problem is primarily directed towards Objectives 9 and 10 (L'Hôpital's rule/mean value theorem and assorted applications of differentiation).

Challenge problem (2 points). Give an example of a function $f(x)$ with exactly two local minima and no global minimum (and show that it has these properties, using the methods we've studied in this unit).

Challenge problem (4 points). At the end of Thursday's lecture, we gave a formula to bound the error in linear approximation

$$e(x) = f(x) - (f(a) - f'(a)(x-a))$$

near some fixed point a, using a bound M on the second derivative $f''(x)$ on a given interval I. Give a similar bound for the error in the *second-order approximation*

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2,$$

using a bound N on the *third* derivative $f'''(x)$ on this interval.

Challenge problem (3 points). A truck driving at v miles per hour has a gas mileage of $\frac{1}{a+bv^2}$ miles per gallon of gasoline, where a and b are some fixed positive constants (depending on the design and fuel efficiency of the truck). The truck driver is paid by the mile, at a rate of r dollars per mile, but has to pay for gasoline, which costs g dollars per gallon.

(a) Assume the driver drives at a constant speed v. Write down an expression $P(v)$ for the total net pay of the driver (i.e. how much they get paid minus gasoline costs) after one hour, in terms of the speed v and the constants a, b, r, and g.

(b) Using your expression from part (a), determine at which speed v the driver should drive in order to maximize their profit if $a = 0.04$ gallons per mile, $b = 0.00001$ gallons $\times$ hours^2 per miles^3, $r = \$0.50$ per mile, and $g = \$4$ per gallon and the speed limit (which the driver does not exceed) is 70 miles per hour.

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	
Challenge problem 1	
Challenge problem 2	
Challenge problem 3	