Lecture 8: product and quotient rules

Calculus I, section 10

September 29, 2022

Review

Last time:

- Definition of derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Linearity

$$\frac{d}{dx}(cf(x)) = cf'(x), \qquad \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

- Power rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

- $\implies$ derivatives of polynomials

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Example: you might be tempted to say $\frac{d}{dx}x^x = x \cdot x^{x-1} = x^x$. This is not true! (The answer is actually $x^x(\log_e(x) + 1)$, which we may encounter later.)

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This is a pain, and with e.g. $f(x) = \frac{x^2+4x-1}{2x^3+7x^2-1}$ will be that much worse.

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First, let's worry about something simpler: products.

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Instead:

Theorem

If f and g are differentiable, then

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

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\begin{aligned}
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&\qquad\qquad \left. + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right) \\
&= \lim_{h\to 0} \left(\frac{f(x+h)(g(x+h) - g(x))}{h} \right. \\
&\qquad\qquad \left. + \frac{(f(x+h) - f(x))g(x)}{h} \right)
\end{aligned}
$$

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We could also do this by linearity:

$$(2x^4 + x^2 + x + 5)(x^2 - x - 1) = 2x^6 - 2x^5 - x^4 + 3x^2 - 6x - 5$$

and proceed as usual term-by-term.

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Using a slightly more complicated argument we can do something similar for fractional powers using the product rule.

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$$f'(x) = 6x^5 \cdot \frac{x^2 + 4x - 2}{x^2 + x + 1} - 3x^6 \cdot \frac{x^2 - 2x - 2}{(x^2 + x + 1)^2}.$$

Derivative of $\sin(x)$

The next class of "nice functions" is trigonometric functions. Let's start with $\sin(x)$:

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$$= \lim_{h\to 0}\sin(x)\cdot\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\cdot\frac{\sin(h)}{h}.$$

So this boils down to these two trigonometric limits we've already computed:

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = \cos'(0) = 0$$

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$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \cos(x)\cdot\frac{\cos(h)-1}{h} - \lim_{h\to 0}\sin(x)\cdot\frac{\sin(h)}{h} \\
&= -\sin(x).
\end{aligned}$$

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Other trigonometric functions

The other three trigonometric functions are $\sec(x) = \frac{1}{\cos(x)}$, $\csc(x) = \frac{1}{\sin(x)}$, and $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$.

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We could use the above plus the quotient rule again. However, it's convenient to introduce another rule as a special case: the reciprocal rule.

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- $\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{1}{\tan(x)} = -\frac{\sec(x)^2}{\tan(x)^2} = -\frac{\frac{1}{\cos(x)^2}}{\frac{\sin(x)^2}{\cos(x)^2}} = -\frac{1}{\sin(x)^2} = -\csc(x)^2$