Lecture 20: applications of integration

Calculus I, section 10
December 1, 2022

Welcome to our last regular lecture of the semester! By now, we have a pretty good understanding of definite and indefinite integrals, the relationship between them, and some techniques to calculate indefinite (and thus definite) integrals. Today, we'll switch focus a little and think about some applications of integrals, now that we can calculate them. (As with techniques of integration, this is only a small taste: calculus 2 or many other math or physics classes, among others, give many more examples of applications.)

We start with a perspective we've already seen: the integral should be thought of as a sort of "cumulative" version of the initial function. For example, if we know that a point is moving along a line with speed $v(t)$, then its position $x(t)$ satisfies

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\, dt,$$

i.e. we can recover the total distance traveled between t_1 and t_2 by the integral of the velocity between them. For example, if $v(t) = v$ is constant, then

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v\, dt = v(t_2 - t_1),$$

not surprisingly. If $t_1 = 0$, $t_2 = 1$, and $v(t) = 1 - 2t$, so that at first the point is moving in the positive direction but then stops and begins moving in the negative direction, we find that

$$x(1) - x(0) = \int_0^1 (1 - 2t)\, dt = (t - t^2)\big|_0^1 = 0 - 0 = 0,$$

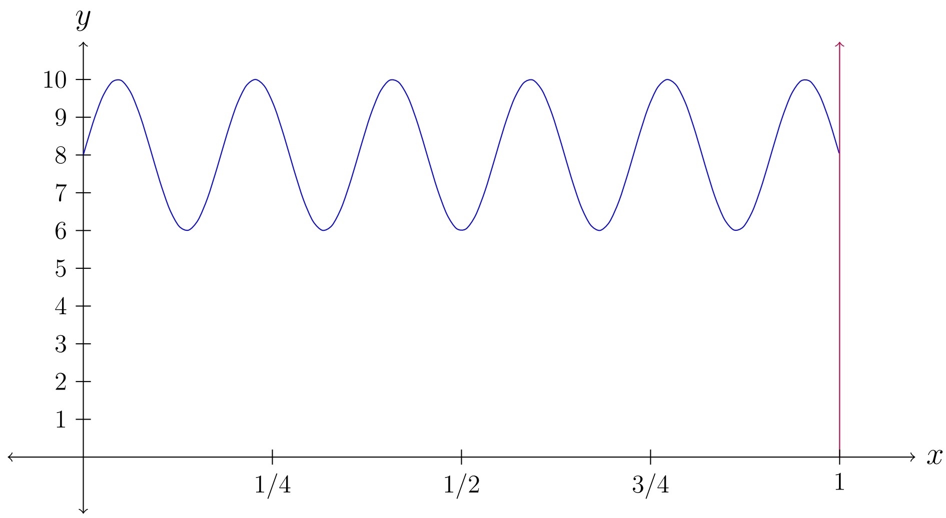
so after 1 second the point has returned to its original position. If we know the initial position $x(0)$, then we could determine $x(1)$ (in this case equal to $x(0)$; more generally, we could find any $x(t)$, as

$$x(t) = x(0) + \int_0^t v(s)\, ds$$

(replacing t by s in the integral since t is already in use).

In physics, it is common to know the acceleration of an object (using Newton's second law, $F = ma$) and want to know its position. Typically, we do so by first finding the velocity from the acceleration and a known initial velocity using the procedure above, and then finding the position from the velocity and a known position similarly. Such a problem is on your homework, so I won't go into too much detail.

Another application related to this perspective of cumulative value is finding the *average* value. Suppose that a runner is running for one hour; sometimes they run faster and sometimes slower, at a speed of $v(t) = 8 + 2\sin(11\pi t)$.



What is the average speed of the runner over the hour?

We don't really have good tools to think about averages of continuously changing quantities. However, as it turns out we don't have to: one simple way of measuring the average speed would be to figure out how far the runner ran would be if we knew how *far* they ran, and then we could simply divide the total distance by the total time (here one hour).

Fortunately, we just saw how to find the total distance they ran:

$$x(1) - x(0) = \int_0^1 v(t)\,dt = \int_0^1 8 + 2\sin(11\pi t)\,dt.$$

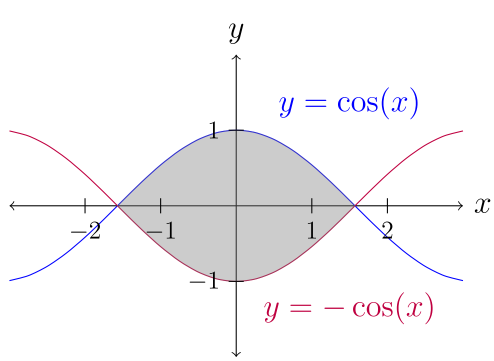
By linearity, this is

$$8 \cdot (1-0) + 2\int_0^1 \sin(11\pi t)\,dt.$$

To conclude, we substitute $u = 11\pi t$, so $du = 11\pi\,dt$, so the integral becomes

$$8 + 2\int_0^{11\pi} \sin(u) \cdot \frac{1}{11\pi}\,du = 8 + \frac{2}{11\pi}\left(-\cos(11\pi) + \cos(0)\right) = 8 + \frac{4}{11\pi} \approx 8.11575.$$

Another common application of integrals is what we originally introduced them for: finding areas. We've looked at finding areas under a given curve, i.e. between the curve and the line $y = 0$. Often, though, we're really interested in finding areas of shapes which aren't easily thought of in this way. For example, how would you find the area of this shape?



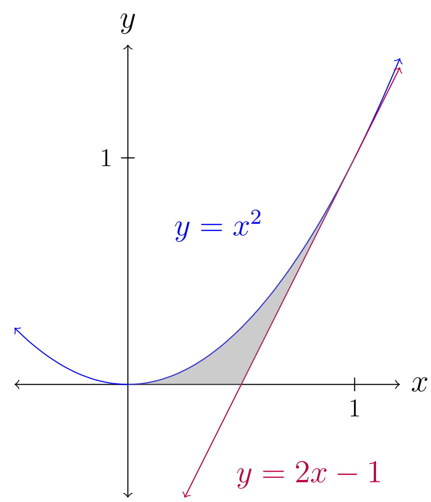
It is not the area beneath a curve; instead, it is the area between two curves!

To solve our problem, we think about integrating the *height* of the region. After all, this is what we're usually doing, it's just that the bottom of the region is usually at 0; this lets us add up the heights as usual to get the total area. Thus whenever we want to find the area between $y = f(x)$ and $y = g(x)$ on a certain region, if say $f(x)$ is on top in that region then we integrate $f(x) - g(x)$. In this case, $\cos(x)$ is greater than $-\cos(x)$ in this region, so we're integrating $\cos(x) - (-\cos(x)) = 2\cos(x)$. (We could also have guessed this by observing that the x-axis divides the region into two identical regions, each of which looks like the integral of $\cos(x)$.)

Finding the bounds can also be a little tricky with these kinds of problems; they're not always given to us. Here, we're going between two points where the curves intersect, which is also at $y = 0$; we have $\cos(x) = -\cos(x) = 0$ at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, and so on, and also at $-\frac{\pi}{2}$, $-\frac{3\pi}{2}$, and so on. Here, the relevant points are $\pm\frac{\pi}{2}$, so our total area will be

$$\int_{-\pi/2}^{\pi/2} 2\cos(x)\,dx = 2\sin(\pi/2) - 2\sin(-\pi/2) = 4.$$

Another example is this: find the area of the shaded region.

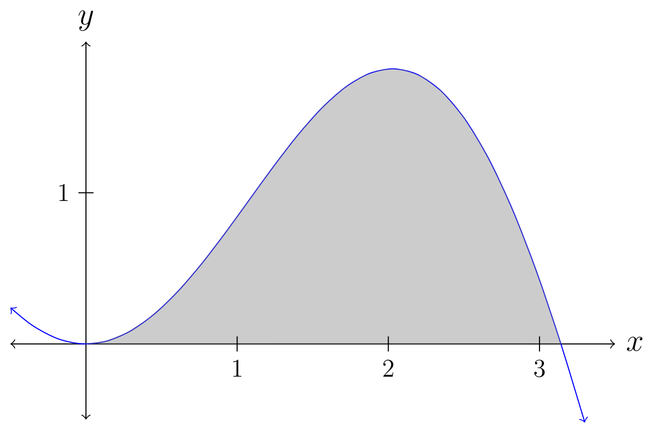


We could try to do the same thing. Now, though, the lower bound is piecewise: up to $x = \frac{1}{2}$ it's just 0, but between $\frac{1}{2}$ and the intersection of $y = x^2$ and $y = 2x - 1$ (which we can find is at $x = 1$) it's $y = 2x - 1$. Thus we need to split the integral: the area is

$$\begin{aligned}
\int_0^{1/2} x^2\,dx + \int_{1/2}^1 x^2 - 2x + 1\,dx &= \frac{1}{3}x^3\Big|_0^{1/2} + \left(\frac{1}{3}x^3 - x^2 + x\right)\Big|_{1/2}^1 \\
&= \frac{1}{3}\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\cdot 1^3 - 1^2 + 1\right) - \left(\frac{1}{3}\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + \frac{1}{2}\right) \\
&= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \\
&= \frac{1}{12}.
\end{aligned}$$

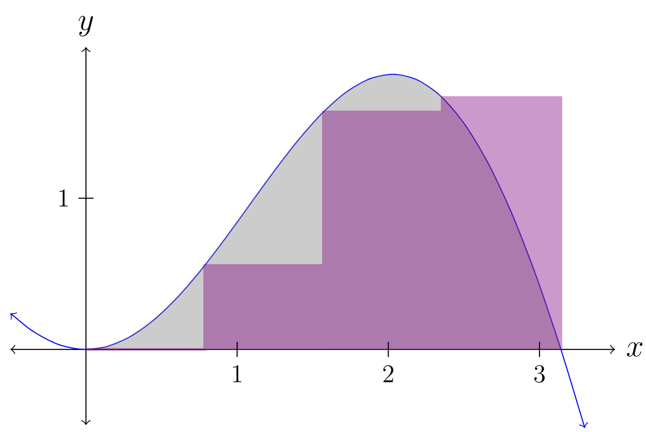
Finally, let's turn to the question of numerical approximation: how does one concretely compute integrals numerically, in cases where there isn't an exact answer or the exact answer is too complicated to be useful? How does your computer or calculator compute integrals?

We've already seen one way to compute integrals: via (either left or right) Riemann sums, which is how we defined them. Let's say we're trying to integrate something like $x\sin(x)$ from 0 to π.

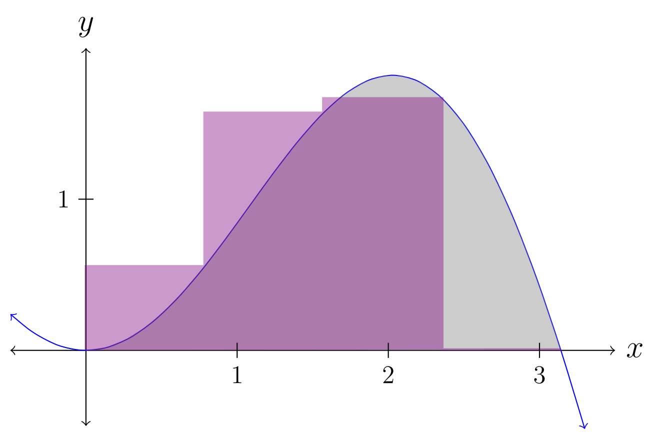


(This is possible to integrate exactly, but it's beyond the methods we've seen in this class; it would be done using integration by parts, and turns out to have value π.)

A four-step left Riemann sum would look like this:

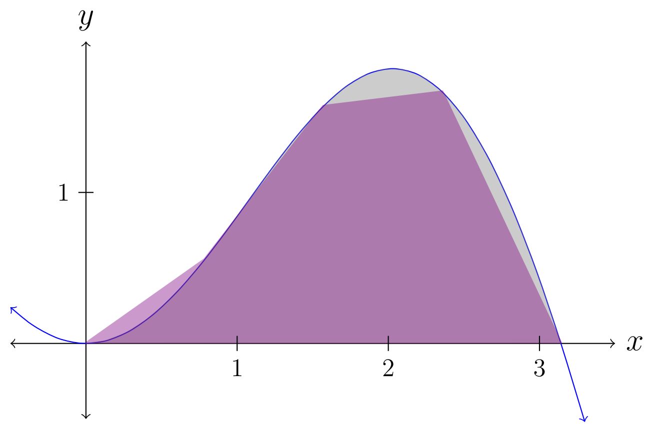


with total area estimate ≈ 2.9784, while the right Riemann sum would look like this:



with the same area estimate (since both endpoints have value 0, so it's just the same area shifted).

For theoretical purposes, this is enough to define integrals, but for practical purposes we might want to do better. As someone suggested a few classes ago, we could do this by using nonzero slopes, i.e. replacing our rectangles with trapezoids. This is called the trapezoid rule.



This is still imperfect, but we can see it looks much closer, even with the same number of divisions. (In this case, as it turns out, it nevertheless actually gives exactly the same area estimate as the Riemann sums! This is somewhat special to this case, but isn't hugely unusual either: even though in each section the trapezoidal area is much closer to the real one, the errors in the Riemann sum method cancel out pretty well, so it's closer than it looks. Nevertheless in general the trapezoidal rule is much more reliable and converges much faster.)

An even more sophisticated approach is via Simpson's rule, which works by approximating the function by a *quadratic* interpolation between the endpoints, taking into account differential data. By using any given number of approximations, one can find convenient formulas; nevertheless in practice this turns out to be computationally more expensive than using the trapezoid rule or similar and just taking more intervals, which usually works better.

Finally, it's worth mentioning adaptive integration. Like for Newton's method, when we're trying to approximate integrals up to a given precision, say using Riemann sums or the trapezoid rule, we can just keep dividing into more and more intervals until the answer seems to be stabilizing: if the answer at N intervals and at $N+1$ intervals agree up to the allowed error, it's probably safe to stop. Adaptive integration takes this idea and goes further with it: in addition to checking the total error, we can also check the error on each interval. Thus if we're doing Riemann sums, regions in which the function is pretty close to constant don't really need very many intervals to get a pretty good approximation; but areas where the function is changing rapidly (i.e. has large derivative, in absolute value) need many more intervals. Similarly for the trapezoid rule, if the function is pretty close to linear the trapezoid rule will do pretty well even on relatively few intervals; but if it's far from linear, as near the maximum of $x\sin(x)$ above, it needs more subdivisions to get a good approximation. In general, we can automate this process—and this is often what computers do—by setting an error threshold for each interval, and dividing each interval only until it reaches that threshold; that way we put our resources where they will be most useful, and get a good approximation fastest.