Lecture 2: more functions

Calculus I, section 10
September 8, 2022

First of all: we may not get through everything today and will go quickly on many things. However, there is another review of precalculus materials being held on Zoom by Professor Sengupta and his TAs Thursday September 8 at 2 and 5 PM with a third session yet to be scheduled. You can also use these notes to review material we don't get to; theoretically, none of this should be new (but might be in practice, every curriculum is different).

1. Examples of functions

Last time, we talked a lot about what a function is and what we're allowed to do with them, but we didn't give a ton of examples. Today is all about examples of functions, how to work with them, and a few more ways of making new functions from old ones. For now, let's assume all our functions are $\mathbb{R} \to \mathbb{R}$, i.e. they take in any real number and spit out another real number.

1.1. Polynomials

The simplest kind of functions are constant ones: $f(x) = c$ for c some fixed number, not depending on x; for example, $f(x) = 5$, $f(x) = 0$, or $f(x) = -1$. If the function we have to work with is of this form, we're happy: our lives are going to be pretty easy.

The next-best thing we could ask for is a function like $f(x) = x$: it takes in a real number x and doesn't do anything to it, just spits out the same thing it got in. This isn't quite as straightforward as a constant function since the output isn't always the same, but it's basically the simplest non-constant function and we're still pretty happy with it.

Now, recall our various ways of making new functions from old ones: since these are functions $\mathbb{R} \to \mathbb{R}$, we can compose them or add, subtract, multiply, or divide them. To avoid having to worry about dividing by zero, let's drop division from the list for now, and composition turns out not to give us anything very interesting with just these functions, but addition, subtraction, and multiplication are fine: for example, using them we can form functions like $f(x) = 3x$, or $f(x) = x - 1$. We can do more than one operation, too, to get things like $f(x) = 4x + 6$.

A slightly more complicated version is when we multiply $f(x) = x$ with *itself*: this gives the function $g(x) = x^2$, which is still not too bad but certainly more complicated than the previous examples. Combining it with x and the constant functions as above gives things like $2x^2 - 3x + 5$; we can also multiply with x more times to get things like x^3, x^4, $x^7 + 2x^3 - 9x - 1$, and so forth. These are *polynomials*: anything that you can make using addition, subtraction, or multiplication from the constant functions together with x. In practice, we can equivalently think of these as functions which can be written as $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ where the a_i are some constants and $n \geq 0$ is some integer, since any expression involving adding, subtracting, or multiplying constants with x can be written in this form: for example, $(x+1)(x^2-2x-2) = x^3 - x^2 - 4x - 2$ by distributing.

Generally speaking, polynomials are very "nice" functions in ways we'll talk more about over the next few weeks. One way in which they're nice is that they're always defined for any real number. This will *not* always be true for our next class of examples.

1.2. Rational functions

Since we formed polynomials using addition, subtraction, and multiplication, it's natural to ask what happens if we add division. The result is the notion of a rational function, i.e. a ratio of two polynomials, for example $\frac{2x^2-3x+1}{x-4}$. These are also nice functions in some ways, but in other ways they are much worse. For example, they are *not* always defined for all real numbers: if the denominator is zero, the function doesn't make sense. For example, the previous function is undefined at $x = 4$, since then the denominator is zero.

The exact ways in which rational functions can fail to be defined are a little bit complicated, and we'll talk more about them when we discuss continuity and asymptotes in a few weeks.

1.3. Exponential functions

In addition to multiplying and adding copies of x (together with constants) some fixed number of times, which gives polynomials, we could also try multiplying constants a variable number of times: for example, $f(x) = 2^x$, so that $f(0) = 2^0 = 1$, $f(1) = 2^1 = 2$, $f(2) = 2^2 = 4$, $f(-1) = 2^{-1} = \frac{1}{2}$, and so on. Let's pause to review exponentials quickly.

In the same way that multiplication can be thought of as repeated addition, exponentiation can be thought of as repeated multiplication: an expression like 2^3 means that we multiply together 3 copies of 2, i.e. $2^3 = 2 \cdot 2 \cdot 2 = 8$. Another important example is that for any number b, b^1 is always just b: it's just one copy of b "multiplied together," so we're not doing anything to it, it's just b again.

There are a few important rules here to be aware of:

- Additivity: $b^{x+y} = b^x \cdot b^y$ for any real numbers b, x, y. (We'll talk about what this means when x and y aren't integers in a minute.) This is because b^x is x copies of b multiplied together, b^y is y copies of b, so multiplying the two of them gives $x + y$ copies of b, i.e. b^{x+y}. For example, $2^3 \cdot 2^1 = (2 \cdot 2 \cdot 2) \cdot (2) = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.

This already gives us a good way of understanding negative exponents, which don't really make sense with the definition above: we can't take -5 copies of 2, but we can still make sense of 2^{-5} because we know that $2^{-5} \cdot 2^6 = 2^{-5+6} = 2^1 = 2$, so $2^{-5} = \frac{2}{2^6} = \frac{2}{64} = \frac{1}{32} = \frac{1}{2^5}$. This works in general: $b^{-x} = \frac{1}{b^x}$. In particular, $b^{-1} = \frac{1}{b}$, and $1 = b^{-1} \cdot b = b^{-1+1} = b^0$, i.e. $b^0 = 1$ for every b.

- Multiplicativity: $(b^x)^y = b^{x \cdot y}$. This is a little bit harder to see: think about the example $(b^2)^3$. This is $b^2 \cdot b^2 \cdot b^2$, but each b^2 is $b \cdot b$, so in total this is $(b \cdot b) \cdot (b \cdot b) \cdot (b \cdot b) = b^6 = b^{2 \cdot 3}$. It's not obvious, but you can do a few more such problems and convince yourself that this works in general.

This lets us define *fractional* exponents. For example, we don't yet know what $b^{1/2}$ is, but whatever it is we know that $(b^{1/2})^2 = b^{\frac{1}{2}\cdot 2} = b^1 = b$, so $b^{1/2}$ is some number whose square is b, i.e. $\sqrt{b}$. (Like with square roots, we use the convention that $b^{1/2}$ is nonnegative.)

Similarly, $b^{1/3} = \sqrt[3]{b}$, and so on: this lets us define $b^{1/n}$ for any positive integer n. Therefore for any positive *rational*[1] number m/n, we can define $b^{m/n} = (b^{1/n})^m$, and so b^x makes sense for any positive rational number x. Using the fact from before that $b^{-1} = \frac{1}{b}$, we get that $b^{-x} = (b^x)^{-1} = \frac{1}{b^x}$, so b^x is defined for all rational x.

You might complain that we want to define it for all *real* x, and as you may or may not know not all real numbers are rational: for example $\sqrt{2}$ or π cannot be written as a fraction of integers. We'll come back to this when we talk about continuity, but the idea is that we can *approximate* any real number by a rational number (for example by a decimal approximation, since e.g. $\pi \approx 3.14159 = \frac{314159}{1000000}$). We can then "fill in the gaps" to define $f(x) = b^x$ for all real x.

One thing to be careful of is that in all the above, we implicitly assumed b was a positive number. If b is negative, some things fail: for example, $(-1)^{1/2} = \sqrt{-1}$ is not a real number. Once we introduce complex numbers, such as $i = \sqrt{-1}$, this problem goes away,[2] but we won't deal with that in this class.

We can build more complicated functions out of exponentials: for example, $f(x) = 5 \cdot 3^x$, or $f(x) = 2^x + 6^x$. We could also combine them with polynomials to get even more complicated things like $2^{-x} + x^2 + 1$ or $x^4 \cdot (\frac{2}{3})^x - 2$.

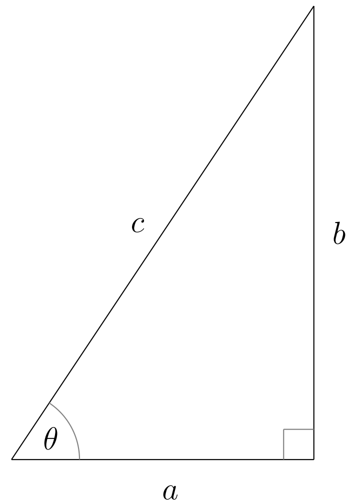
Exponentials are also fairly nice functions, though in some ways (such as the complicated way we have to build them as above) less nice than polynomials.

1.4. Trigonometric functions

Another class of functions is trigonometric functions. These have to do with triangles or the unit circle, depending how you prefer to think of them. One way is like this: suppose we have a right triangle with side lengths a, b, and c, and call the angle shown θ.

[1] A rational number is just one that can be written as a fraction with the numerator and denominator both integers. This includes all integers, since we can write for example -3 as $\frac{-3}{1}$, but it also includes for example $\frac{2}{3}$ or $-\frac{16}{5}$.

[2] Really it gets transformed into a different problem; complex analysis is tricky that way.



Up to rescaling the whole thing, the shape of the triangle is completely governed by θ; one way to see this is that the angles of a triangle have to add up to 180°, so since one is 90° and another is θ the last one has to be $90^\circ - \theta$.

Therefore we'd like to be able to, given θ, say what the side lengths are; since they're only determined up to scaling, the best we can hope to do is say what the ratios are. These are named as follows:

$$\cos\theta = \frac{a}{c}$$
$$\sin\theta = \frac{b}{c}$$
$$\tan\theta = \frac{b}{a}.$$

These are short for cosine, sine, and tangent. (In fact, we're making another abbreviation, which is writing $\sin\theta$ for $\sin(\theta)$ and so on; this is generally harmless and saves a little space, but when you're taking the sine or cosine of more complicated things it's safest to put in the parentheses.)

There are three other possible ratios, which also have names, but we usually use the three above since the remaining three are just their inverses; they are

$$\sec\theta = \frac{c}{a} = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{c}{b} = \frac{1}{\sin\theta}$$
$$\cot\theta = \frac{a}{b} = \frac{1}{\tan\theta},$$

short for secant, cosecant, and cotangent.[3]

[3]There are geometric reasons behind these names, which we won't get into.

There are many relationships between these functions which can be derived from the picture above. The most important of these is the relation

$$\sin^2\theta + \cos^2\theta = 1,$$

where $\sin^2\theta$ is short for $(\sin\theta)^2$ and similarly for $\cos^2\theta$. This comes from the Pythagorean theorem: $a^2+b^2=c^2$, so dividing by c^2 we get $\frac{a^2}{c^2}+\frac{b^2}{c^2} = (\frac{a}{c})^2+(\frac{b}{c})^2 = \cos^2\theta+\sin^2\theta = \frac{c^2}{c^2} = 1$. Many other relations can be derived from this one.

Another important relationship is that $\cos(90^\circ-\theta) = \sin(\theta)$, and vice-versa $\sin(90^\circ-\theta) = \cos\theta$. We can see this by flipping the triangle: as mentioned above, $90^\circ - \theta$ is the angle opposite θ, and using that angle instead swaps a and b, and therefore swaps sine and cosine.

Finally, a word on units: we can talk about angles either using degrees, as we have been so far, or radians. For degrees, the whole circle is 360°, so a right angle, which is a quarter of that, is 90°; for radians, a circle is 2π radians, since that's the circumference of a circle of radius 1, and so a right angle is $\frac{2\pi}{4} = \frac{\pi}{2}$. You can convert between them using this relationship, or the equivalent relationship that 180° is π radians: for example, 30° is $\frac{1}{6}$ of 180° and so is $\frac{\pi}{6}$. Going forward, we'll mostly use radians in this class, but feel free to convert to degrees and back if you find that easier; answers in degrees are also fine, you just need to be able to understand a problem given in radians.

2. Inverse functions

2.1. Inverse functions

Suppose we're given a relationship between two variables $y = f(x)$, i.e. y depends on x in some way given by a function of x. If we know y, can we recover x?

Sometimes, the answer is yes. For example (this problem was on the survey), if $y = f(x) = 2x+1$, then we can just solve for x:

$$x = \frac{y-1}{2} = \frac{y}{2} - \frac{1}{2}.$$

In this case we say that f is *invertible*, and write its inverse as f^{-1}, so in this case we have $f^{-1}(x) = \frac{x-1}{2}$ and, with the relationship above, $x = f^{-1}(y)$ is equivalent to $y = f(x)$.

More precisely, f^{-1} is a function such that $f^{-1}(f(x)) = x$ for every x in the domain of f, *and* $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}. In the case above, this means all real numbers on both sides, and we can check that this is true:

$$f(f^{-1}(x)) = f\left(\frac{x-1}{2}\right) = 2\cdot\frac{x-1}{2} + 1 = x-1+1 = x,$$

and

$$f^{-1}(f(x)) = f^{-1}(2x+1) = \frac{(2x+1)-1}{2} = \frac{2x}{2} = x.$$

Generally, the method for finding inverses of functions is exactly like above: set $y = f(x)$ and solve for x. If you get a function of y, that's the answer. A slightly more complicated example is something like $y = f(x) = x^3 - 1$; solving for x, we get $x = f^{-1}(y) = \sqrt[3]{y+1}$.

Some functions, however, don't have inverses. A simple example is $f(x) = x^2$. If we try to solve $y = x^2$ for x, we get two possible solutions: $x = \pm\sqrt{y}$. This isn't a function, because there are inputs for y for which it outputs two different numbers (namely all $y > 0$), and there are inputs for which it isn't defined (all $y < 0$).

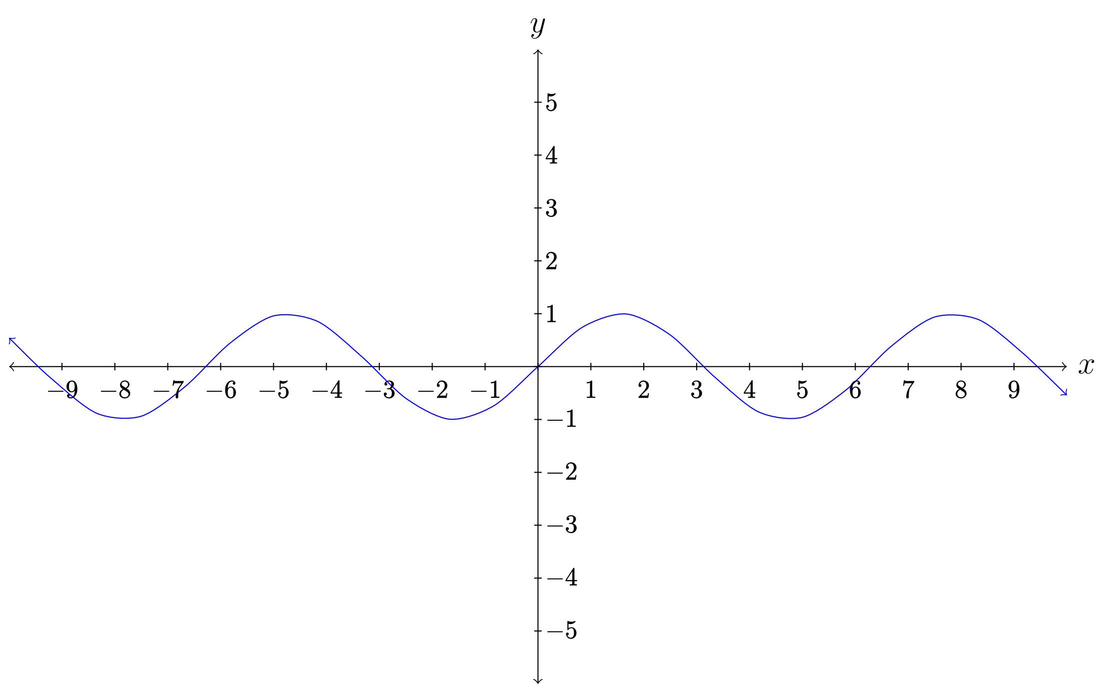
What went wrong? Two things: first, there are values of y for which there is more than one x with $f(x) = y$. Second, there are values of y for which there is *no* x with $f(x) = y$. These give rise to the two kinds of problematic y above.

This suggests the following: a function is invertible if it is one-to-one, i.e. for every y there is at most one x with $f(x) = y$, and onto, i.e. for every y there is at least one x with $f(x) = y$; in other words, there is always exactly one such x. The function $f(x) = 2x + 1$ has this property, but as we've seen $f(x) = x^2$ does not.

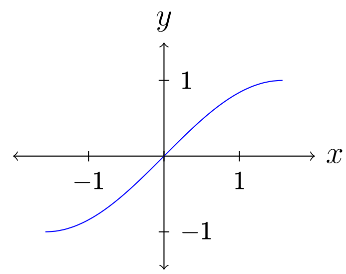
One way of thinking of this test is called the "horizontal line test": if you graph your function and draw a horizontal line across the graph at any point, you should hit exactly one point on the graph. If you can choose your line such that you hit no points or more than one point, the function is not invertible.

One way of fixing the problem when a function is not invertible is by restricting the domain and codomain; this is where the formal perspective from last time comes in handy. For example, with we think of $f(x) = x^2$ as a function $\mathbb{R} \to \mathbb{R}$, it is not invertible, because it is neither one-to-one nor onto. If we restrict the domain to nonnegative numbers, though, it becomes one-to-one: for any y, there is at most one *nonnegative* x such that $x^2 = y$. It still isn't onto, but we can fix this too: every *nonnegative* y has a square root, so $f(x) = x^2$ *is* invertible as a function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

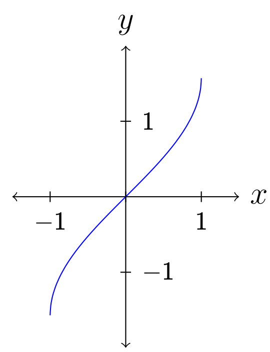
Another important example is for trigonometric functions. Consider the function $y = \sin x$.



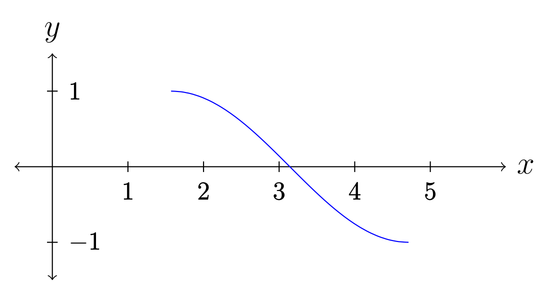
This clearly does not pass the horizontal line test: it is neither one-to-one nor onto. However, we can do the same restricting the domain and codomain trick. Restricting the codomain is pretty straightforward: the image is the set of numbers between -1 and 1, written as $[-1, 1]$. Restricting the image is a little more tricky: there are many different ways we can do it. For example, we could choose the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$:



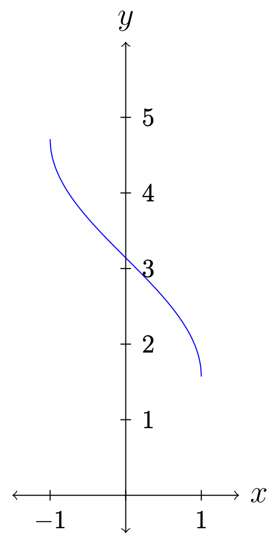
Now this is invertible! Its inverse looks like this:



But we could also have chosen for example $[\frac{\pi}{2}, \frac{3\pi}{2}]$, in which case the graph would look like this:



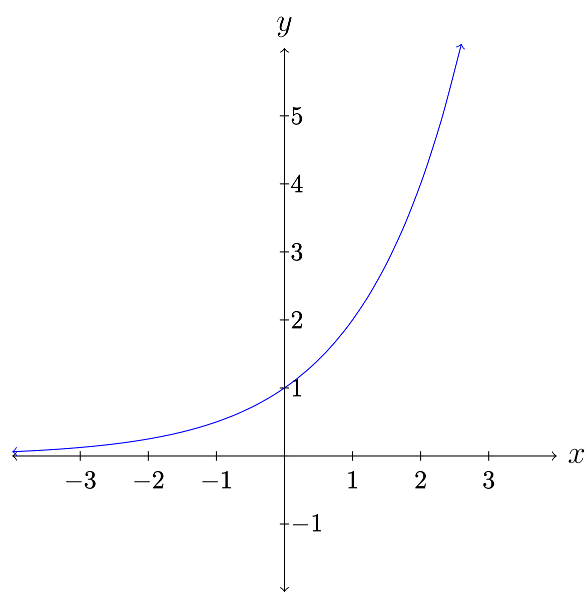
and this is again invertible, with inverse function looking as follows:



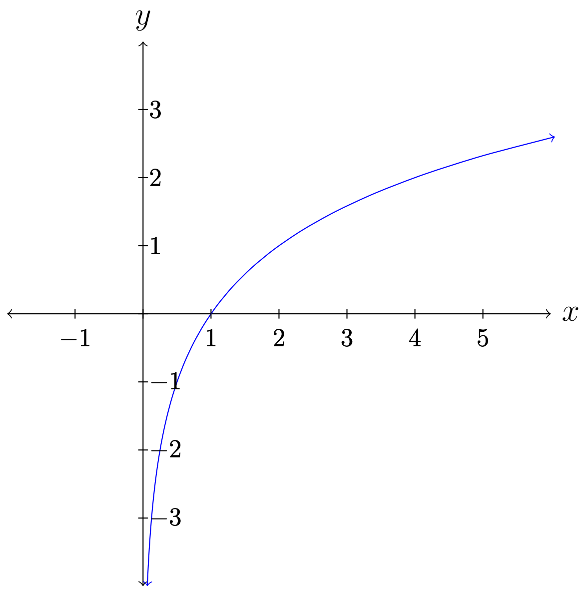
2.2. Logarithms

Another special kind of inverse function is worth spending some time on: inverse functions of exponentials. Namely, if $f(x) = b^x$ for some fixed $b > 0$, does it have an inverse function?

Let's try plotting an example: $f(x) = 2^x$ looks like this.



We'll never be able to intersect more than one point by drawing a horizontal line, but sometimes there will be no points, so strictly speaking as a function $\mathbb{R} \to \mathbb{R}$ this is not invertible. However, 2^x will always be positive for any real x (since if $x \geq 0$, 2^x is positive, and $2^{-x} = \frac{1}{2^x}$ is also positive), and if we restrict to codomain $\mathbb{R}_{>0}$ then this becomes invertible. Its inverse has no simple description other than "the inverse function of 2^x"; instead we give it a name, $\log_2(x)$, read as the logarithm of x with base 2. It is a function $\mathbb{R}_{>0} \to \mathbb{R}$, i.e. it takes in a *positive* real number and spits out any real number; its defining property is that $\log_2(x)$ is the number such that $2^{\log_2(x)} = x$, and similarly $\log_2(2^x) = x$. Its graph looks like this:



For example, $\log_2(8) = 3$, since $2^3 = 8$; $\log_2(1) = 0$, since $2^0 = 1$.

More generally, we define $\log_b(x)$ to be the inverse function of b^x for every positive number b.

We know some important properties of exponential functions, and we can translate these into properties of logarithms. For example, we know that $b^{x+y} = b^x \cdot b^y$. If we replace x by $\log_b(x)$ and y by $\log_b(y)$, this means that

$$b^{\log_b(x)+\log_b(y)} = b^{\log_b(x)} \cdot b^{\log_b(y)} = x \cdot y.$$

Taking logarithms of both sides, we get

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y).$$

A special case of this formula gives

$$\log_b(x^2) = \log_b(x \cdot x) = 2\log_b(x).$$

We might guess in general that we have

$$\log_b(x^y) = y \cdot \log_b(x),$$

and indeed this is true: you can see it from the multiplicative property of exponentials, as follows. We have $x^y = (b^{\log_b(x)})^y = b^{y \cdot \log_b(x)}$, so taking $\log_b$ of both sides gives $\log_b(x^y) = \log_b(b^{y \cdot \log_b(x)}) = y \cdot \log_b(x)$.

Finally, we can relate logarithms of different bases, as follows. Fix two positive numbers b and c, so that $\log_b(x)$ is the inverse function of b^x and $\log_c(x)$ is the inverse function of

c^x. We have $c = b^{\log_b(c)}$ and so $x = c^{\log_c(x)} = (b^{\log_b(c)})^{\log_c(x)} = b^{\log_b(c)\cdot\log_c(x)}$, so $\log_b(c) \cdot \log_c(x)$ is a number y such that $b^y = x$. But this is the definition of $\log_b(x)$! So we conclude that $\log_b(x) = \log_b(c) \cdot \log_c(x)$, or in other words

$$\log_c(x) = \frac{\log_b(x)}{\log_b(c)}.$$

This lets us convert from one base to another.