# Lecture 16: graphs and Newton's method

Calculus I, section 10
November 10, 2022

Today we'll complete our unit on applications of derivatives, with some observations on the relationship between graphs and derivatives and a numerical technique to solve equations using differentiation.

These days, it is rarely necessary to draw graphs by hand: a graphing calculator can do it, or a website even better and more efficiently. Nevertheless although you'll rarely need to actually draw graphs, it's still useful to be able to see roughly what the graph of a function should look like to better understand the function and the relationship between its graphical and algebraic interpretations.

Precise graphing is best left to computers, but we can get a good sense of what the graph of $y = f(x)$ should look like based on knowing the following:

- where $f(x)$ is positive or negative and where it crosses the $x$-axis, i.e. where $f(x) = 0$;
- where $f(x)$ is increasing or decreasing (i.e. the sign of $f'(x)$) and where it changes, i.e. where $f'(x) = 0$;
- where $f(x)$ is concave up or down, i.e. whether $f'(x)$ is increasing or decreasing, i.e. the sign of $f''(x)$;
- the end behavior of $f(x)$, i.e. what happens to $f(x)$ as $x \to \pm\infty$;
- asymptotes of $f(x)$, i.e. points $x$ at which $f(x) \to \pm\infty$;
- various extra information, such as:
  - $f(0)$
  - jumps in $f(x)$
  - endpoints
  - symmetries of $f(x)$, such as being periodic or even/odd.

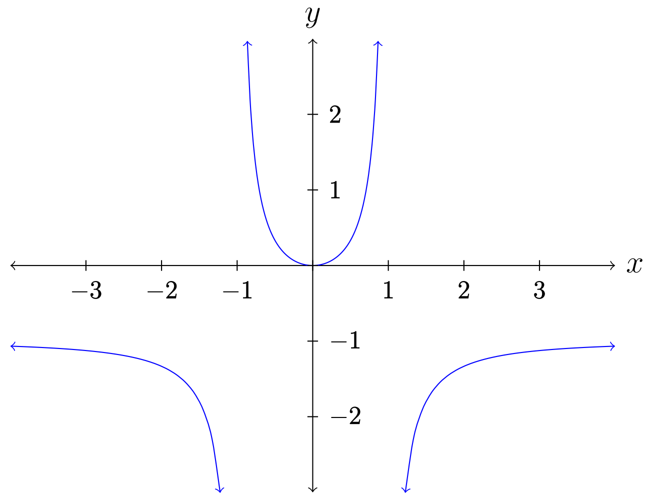
Let's try an example: take $f(x) = \frac{x^2}{1-x^2}$. We have:

- $x^2$ is always positive and $1 - x^2$ is positive for $-1 < x < 1$ and negative for $x > 1$ or $x < -1$, so $f(x)$ is positive for $-1 < x < 1$ and negative elsewhere;
- $f'(x) = \frac{2x}{(1-x^2)^2}$ is positive for $x > 0$ and negative for $x < 0$;
- $f''(x) = \frac{2(3x^2+1)}{(1-x^2)^3}$ has the same sign as $f(x)$, i.e. positive for $-1 < x < 1$ and negative elsewhere;
- as $x \to \pm\infty$, we have $f(x) = \frac{1}{1-x^2} - 1 \to -1$, so there is a horizontal asymptote at $y = -1$ in both directions;
- $f(x) = \frac{x^2}{(1+x)(1-x)}$ has vertical asymptotes at $x = \pm 1$;
- there are no jumps or endpoints, and $f(0) = 0$.

What about the symmetry question? Being periodic means that there is some number $P$ such that for every $x$, we have $f(x+P) = f(x)$. For example, $\sin(x)$ and $\cos(x)$ are both periodic with period $P = 2\pi$; $\tan(x)$ is periodic with period $\pi$.

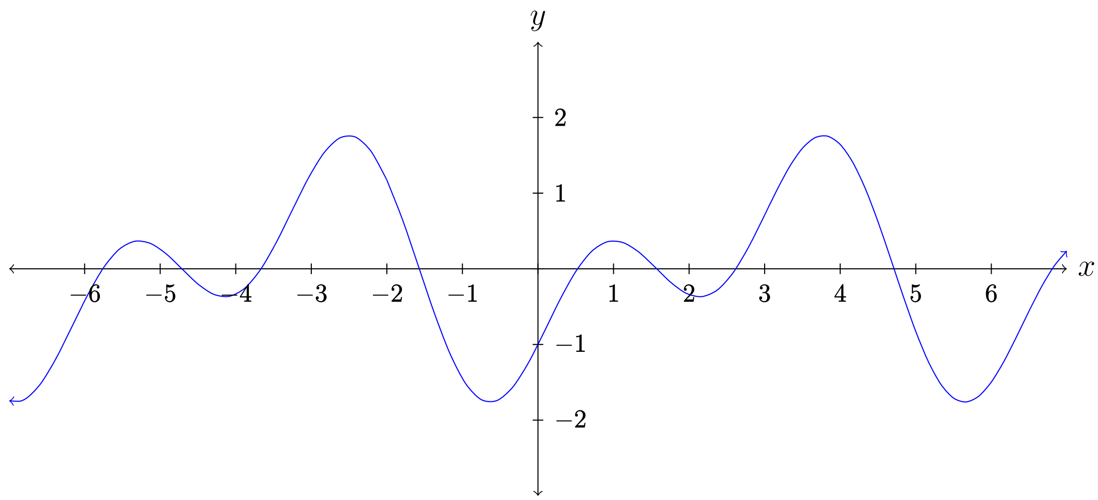
In this case, $f(x)$ is not periodic. It does have a different kind of symmetry, though: $f(-x) = f(x)$ for every $x$. This is called being even; we've seen this behavior before, in $\cos(x)$, since $\cos(-x) = \cos(x)$. Another kind of behavior is possible: $\sin(-x) = -\sin(x)$, and more generally if $f(-x) = -f(x)$ for every $x$ we say that $f(x)$ is odd.

In our case since $f(x)$ is even, we know that it is symmetric under reflection across the $y$-axis. We have enough information to draw a pretty good graph:




Another example is $f(x) = \sin(2x) - \cos(x)$.

- Let's first think about the range $-\pi \leq x \leq \pi$ (the reason why will become clear later). We have $\sin(2x) = 2\sin(x)\cos(x)$, so $f(x) = (2\sin(x) - 1)\cos(x) = 0$ means that either $\sin(x) = \frac{1}{2}$ (which occurs in this range at $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$) or $\cos(x) = 0$ (at $x = \frac{\pi}{2}$ or $x = -\frac{\pi}{2}$). Thus we can check that $f(x)$ is positive between $-\pi$ and $-\frac{\pi}{2}$, negative between $-\frac{\pi}{2}$ and $\frac{\pi}{6}$, positive between $\frac{\pi}{6}$ and $\frac{\pi}{2}$, negative between $\frac{\pi}{2}$ and $\frac{5\pi}{6}$, and positive between $\frac{5\pi}{6}$ and $\pi$.
- $f'(x) = 2\cos(2x) + \sin(x)$ is a little tricky to find the zeros of, but to get a rough idea of what the graph should look like it's enough to observe that at $-\pi$, it is positive (equal to 2); at $-\frac{\pi}{2}$, it is negative (equal to $-1$); at 0 it is positive (equal to 2); at $\frac{\pi}{2}$ it is negative (equal to $-1$); and at $\pi$ it is positive again (equal to 2).
- The first derivative was hard enough to assess, so we skip the second derivative.
- There are no vertical asymptotes or jumps.
- $f(x)$ is periodic, with period $2\pi$! Therefore whatever happens in this region tells us what happens everywhere, and so our graph looks something like this:

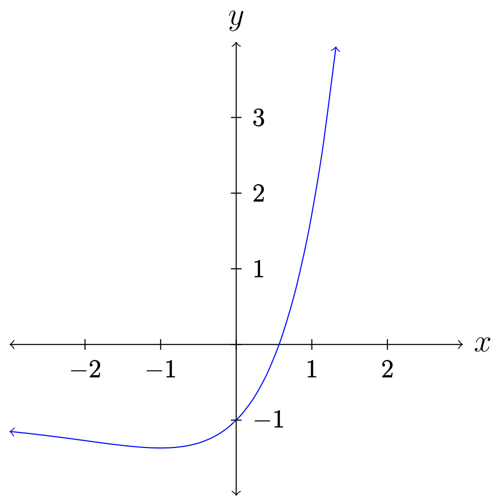



Before we move on to Newton's method, let me mention some transformations we sometimes like to do to graphs and how we could think of them algebraically. One is that we sometimes would like to move the graph around. Vertically, this is straightforward: to move the graph up by $a$ units, we just take $y = f(x) + a$. Horizontally, it's a little more confusing: to move the graph to the right (in the positive direction) by $a$ units, we would take $y = f(x - a)$. This is so that e.g. at the point $x = a$, in the new version we now get what would have previously been the value at 0, i.e. $f(a - a) = f(0)$.

Another transformation is scaling. We can scale vertically by $y = af(x)$, or horizontally by $y = f(x/a)$ (again note the expected direction is reversed, for the same reason). To do both simultaneously, preserving the aspect ratio, we would take $y = af(x/a)$.

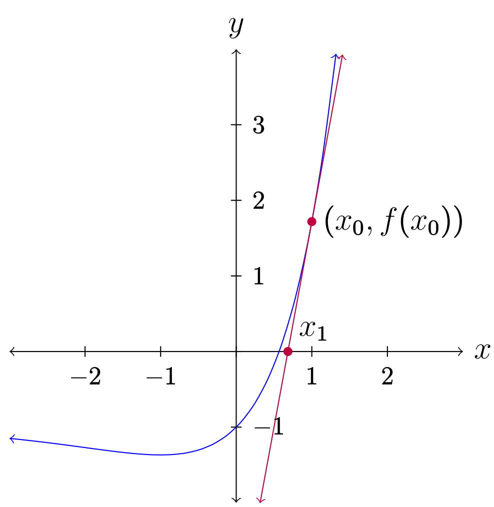
We now move on to a completely separate application, and one which again has the mysterious quality that it does not initially seem related to derivatives at all. Say we have some equation we want to solve, $f(x) = 0$; for a concrete example, take $f(x) = xe^x - 1 = 0$. Although it looks simple, this equation has no algebraic solution: we cannot write down a formula for the solution $x$ using the operations we know! This is a frequent occurrence; sometimes even when we can write down an answer, it is simply too complicated to work with in practice. For example, $x^4 + 2x^3 - x^2 - 1 = 0$ has a unique positive real solution, which admits an exact formula involving roots which is nevertheless too complicated to write down on the blackboard (you can look it up if you so desire—it's even too complicated for me to copy here). However it is much easier to work with its numerical approximation $x \approx 0.8445$, which for many purposes is sufficient.

How can we numerically approximate solutions? The trick is as follows. Take the example above, $f(x) = xe^x - 1$. The graph looks like this:




We're looking for the point where the curve crosses the $x$-axis.

Let's start by making a wild guess, which we call $x_0$; say $x_0 = 1$, which looks like it's probably not too far from the right answer. The idea is that by looking at the derivative, we can see in which direction we should change our guess in order to move towards the correct value. In particular, if we imagine that the true value $x$ is close enough to $x_0$ that it is reasonable to linearly approximate $f$ at $x$ using the derivative at $x_0$, then we have $f(x) = 0 \approx f(x_0) + f'(x_0) \cdot (x - x_0)$. Of course, we don't know $x$, but if we imagine that this approximation is actually an equation then we could solve for it: we get $x \approx x_1 := x_0 - \frac{f(x_0)}{f'(x_0)}$. This is the point at which the line tangent to $y = f(x)$ at $x_0$ would intersect the $x$-axis.




We can then repeat the process, with our new guess of $x_1$, to get $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, and so on. This is called Newton's method; it is the beginning of a whole field of numerical analysis.

In this case, we know $f(x) = xe^x - 1$ and $f'(x) = (x+1)e^x$, so we know everything we need to start computing. With $x_0 = 1$, we get

$$\begin{aligned}
x_1 &\approx 0.6839397, \\
x_2 &\approx 0.5774545, \\
x_3 &\approx 0.5672297, \\
x_4 &\approx 0.5671433, \\
x_5 &\approx 0.5671433,
\end{aligned}$$

and so on—after only a few steps we've converged to a value up to 7 decimal places, and one can check that plugging in these seven places gives $f(0.5671433) \approx 0.0000000265$, already very close to zero.

We could use the same technique on the other example mentioned: $f(x) = x^4 + 2x^3 - x^2 - 1 = 0$. Since we're looking for a positive solution, let's start by guessing $x_0 = 2$. We have $f'(x) = 4x^3 + 6x^2 - 2x$ and so

$$\begin{aligned}
x_1 &= 2 - \frac{f(2)}{f'(2)} = 2 - \frac{27}{52} \approx 1.48077, \\
x_2 &= 1.48077 - \frac{f(1.48077)}{f'(1.48077)} \approx 1.13098, \\
x_3 &= 1.13098 - \frac{f(1.13098)}{f'(1.13098)} = 0.93005, \\
x_4 &= 0.93005 - \frac{f(0.93005)}{f'(0.93005)} \approx 0.85488, \\
x_5 &= 0.85488 - \frac{f(0.85488)}{f'(0.85488)} \approx 0.84468, \\
x_6 &= 0.84468 - \frac{f(0.84468)}{f'(0.84468)} \approx 0.8445,
\end{aligned}$$

so we've gotten there after 6 iterations (and would need another to see that these digits aren't changing). Of course, we could keep going to get more and more accuracy.

Why did it take longer this time even though it seems like a simpler function? Because the point we started with was further away, so several iterations are spent just getting close enough to start getting accuracy. Even so, we get pretty good accuracy within a few iterations.

This is hinting at an important point: Newton's method is highly dependent on the initial choice of point. You can imagine that in our second example, if we'd chosen a different starting point we might have gotten a different solution of the equation. More concerningly, in our first example a poor choice of point would mean Newton's method would never converge!

Let's see what happens if we tried taking $x_0 = -1$ in our first example. We get $f(x_0) = -\frac{1}{e} - 1$ and $f'(x_0) = 0$, so Newton's method cannot even get started! This will always happen if we choose $x_0$ with $f'(x_0) = 0$, so that's a situation we want to make sure to avoid.

Okay, what about $x_0 = -2$? Then we have

$$x_1 \approx -11.3891,$$
$$x_2 \approx -8516.92,$$

and so forth—the $x_n$ are rapidly diverging to $-\infty$! This is picking up on the fact that $f(x)$ does increase, as if moving towards zero, as $x \to -\infty$—but in fact not only does it never get towards zero, but it is not even asymptotic to it, as $\lim_{x\to-\infty} f(x) = -1$. (We could see this by L'Hôpital's rule: $xe^x - 1 = \frac{x-e^{-x}}{e^{-x}}$, whose limit is then the same as that of $\frac{1-e^{-x}}{e^{-x}} = e^x - 1$, which tends to $-1$ as $x \to -\infty$.) In short: Newton's method is very powerful, but also very easy to fool; it only works well if you have a reasonably well-behaved function and make a good choice of initial guess.

One interesting diversion: what would happen if we apply Newton's method to $f(x) = x^2 + 1$, which has no real roots?

We don't want to start at $x_0 = 0$, since there $f'(0) = 0$, so let's start at $x_0 = 1$. Then we find

$$x_1 = 1 - \frac{1^2+1}{2\cdot 1} = 0,$$

so we have to stop there—we know we can't plug in 0 to Newton's method. We might guess that this sort of thing always happens. But in fact reality is weirder: if we start with $x_0 = 2$, then we get

$$\begin{aligned}
x_1 &= 2 - \frac{2^2+1}{2\cdot 2} = \frac{3}{4},\\
x_2 &= \frac{3}{4} - \frac{(3/4)^2+1}{2\cdot\frac{3}{4}} = -\frac{7}{2}4 \approx -0.29,\\
x_3 &\approx 1.57,\\
x_4 &\approx 0.47,\\
x_5 &\approx -0.83,\\
x_6 &\approx 0.19,\\
x_7 &\approx -2.54,
\end{aligned}$$

and so on—bouncing around seemingly at random and never converging anywhere. This is an example of a *chaotic system*: its behavior is highly sensitive to the initial choice of starting point, and in this case will either blow up after a few iterations or oscillate apparently forever. In fact it is possible to derive an exact formula for these values via some trigonometric identities: if $x_0 = \cot\theta$, then $x_n = \cot(2^n\theta)$. Once $n$ is reasonably large, this oscillates very rapidly in the initial choice $\theta$, so it is not surprising that this gives a chaotic system.