MAPPING CLASS GROUP FACTORIZATIONS AND SYMPLECTIC 4-MANIFOLDS: SOME OPEN PROBLEMS

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Abstract. Lefschetz fibrations and their monodromy establish a bridge between the world of symplectic 4-manifolds and that of factorizations in mapping class groups. We outline various open problems about mapping class group factorizations which translate to topological questions and conjectures about symplectic 4-manifolds.

1. Lefschetz fibrations and symplectic 4-manifolds

Definition 1. *A Lefschetz fibration on an oriented compact smooth 4-manifold M is a smooth map $f : M \to S^2$ which is a submersion everywhere except at finitely many non-degenerate critical points $p_1, \dots, p_r$, near which f identifies in local orientation-preserving complex coordinates with the model map $(z_1, z_2) \mapsto z_1^2 + z_2^2$.*

The fibers of a Lefschetz fibration f are compact oriented surfaces, smooth except for finitely many of them. The fiber through p_i presents a transverse double point, or *node*, at p_i. Without loss of generality, we can assume after perturbing f slightly that the critical values $q_i = f(p_i)$ are all distinct. Fix a reference point q_* in $S^2 \setminus \mathrm{crit}(f)$, and let $\Sigma = f^{-1}(q_*)$ be the corresponding fiber. Then we can consider the *monodromy homomorphism*

$$\psi : \pi_1(S^2 \setminus \mathrm{crit}(f), q_*) \to \mathrm{Map}(\Sigma),$$

where $\mathrm{Map}(\Sigma) = \pi_0\mathrm{Diff}^+(\Sigma)$ is the mapping class group of Σ. The image $\psi(\gamma)$ of a loop $\gamma \subset S^2 \setminus \mathrm{crit}(f)$ is the isotopy class of the diffeomorphism of Σ induced by parallel transport (with respect to an arbitrary horizontal distribution) along the loop γ; in other terms, $\psi(\gamma)$ is the monodromy of the restriction of f to the preimage of γ.

The singular fibers of f are obtained from the nearby smooth fibers by collapsing a simple closed loop, called the *vanishing cycle*. This can be seen on the local model $(z_1, z_2) \mapsto z_1^2 + z_2^2$, whose singular fiber $\Sigma_0 = \{z_1^2 + z_2^2 = 0\}$ is obtained from the smooth fibers $\Sigma_\epsilon = \{z_1^2 + z_2^2 = \epsilon\}$ $(\epsilon > 0)$ by collapsing the embedded loops $\{(x_1, x_2) \in \mathbb{R}^2,\ x_1^2 + x_2^2 = \epsilon\} = \Sigma_\epsilon \cap \mathbb{R}^2$.

The monodromy of a Lefschetz fibration around a singular fiber is the positive Dehn twist along the corresponding vanishing cycle. Choose an ordered collection $\eta_1, \dots, \eta_r$ of arcs joining q_* to the various critical values of f, and thicken them to obtain closed loops $\gamma_1, \dots, \gamma_r$ based at q_* in $S^2 \setminus \mathrm{crit}(f)$, such that each γ_i encircles exactly one of the critical values of

f, and $\pi_1(S^2\setminus\mathrm{crit}(f),q_*)=\langle\gamma_1,\dots,\gamma_r\mid\prod\gamma_i=1\rangle$. Then the monodromy of f along each γ_i is a positive Dehn twist τ_i along an embedded loop $\delta_i\subset\Sigma$, obtained by parallel transport along η_i of the vanishing cycle at the critical point p_i, and in $\mathrm{Map}(\Sigma)$ we have the relation $\tau_1\dots\tau_r=\mathrm{Id}$.

Hence, to every Lefschetz fibration we can associate a *factorization* of the identity element as a product of positive Dehn twists in the mapping class group of the fiber (a factorization of the identity is simply an ordered tuple of Dehn twists whose product is equal to Id; we will often use the multiplicative notation, with the understanding that what is important is not the product of the factors but rather the factors themselves).

Given the collection of Dehn twists $\tau_1,\dots,\tau_r$ we can reconstruct the Lefschetz fibration f above a large disc Δ containing all the critical values, by starting from $\Sigma\times D^2$ and adding handles as specified by the vanishing cycles [13]. To recover the 4-manifold M we need to glue $f^{-1}(\Delta)$ and the trivial fibration $f^{-1}(S^2\setminus\Delta)=\Sigma\times D^2$ along their common boundary, in a manner compatible with the fibration structures. In general this gluing involves the choice of an element in $\pi_1\mathrm{Diff}^+(\Sigma)$; however the diffeomorphism group is simply connected if the genus of Σ is at least 2, and in that case the factorization $\tau_1\dots\tau_r=\mathrm{Id}$ determines the Lefschetz fibration $f:M\to S^2$ completely (up to isotopy).

The monodromy factorization $\tau_1\dots\tau_r=\mathrm{Id}$ depends not only on the topology of f, but also on the choice of an ordered collection $\gamma_1,\dots,\gamma_r$ of generators of $\pi_1(S^2\setminus\mathrm{crit}(f),q_*)$; the braid group B_r acts transitively on the set of all such ordered collections, by *Hurwitz moves*. The equivalence relation induced by this action on the set of mapping class group factorizations is generated by

$$(\tau_1,\dots,\tau_i,\tau_{i+1},\dots,\tau_r)\sim(\tau_1,\dots,\tau_i\tau_{i+1}\tau_i^{-1},\tau_i,\dots,\tau_r)\quad\forall 1\le i<r,$$

and is called *Hurwitz equivalence*. Additionally, in order to remove the dependence on the choice of the reference fiber Σ, we should view the Dehn twists τ_i as elements of the mapping class group Map_g of an abstract surface of genus $g=g(\Sigma)$. This requires the choice of an identification diffeomorphism, and introduces another equivalence relation on the set of mapping class group factorizations: *global conjugation*,

$$(\tau_1,\dots,\tau_r)\sim(\gamma\tau_1\gamma^{-1},\dots,\gamma\tau_r\gamma^{-1})\quad\forall\gamma\in\mathrm{Map}_g.$$

Proposition 2. *For $g\ge 2$, there is a one to one correspondence between* (*a*) *factorizations of* Id *as a product of positive Dehn twists in* Map_g, *up to Hurwitz equivalence and global conjugation, and* (*b*) *genus g Lefschetz fibrations over S^2, up to isotopy.*

The main motivation to study Lefschetz fibrations is that they seem to provide a manageable approach to the topology of symplectic 4-manifolds.

It is a classical result of Thurston that, if M is an oriented surface bundle over an oriented surface, then M is a symplectic 4-manifold, at least provided that the homology class of the fiber is nonzero in $H_2(M,\mathbb{R})$. As shown by

Gompf, the argument extends to the case of Lefschetz fibrations (Theorem 10.2.18 in [12]):

Theorem 3 (Gompf)**.** *Let $f : M \to S^2$ be a Lefschetz fibration, and assume that the fiber represents a nonzero class in $H_2(M, \mathbb{R})$. Then M admits a symplectic structure for which the fibers of f are symplectic submanifolds; this symplectic structure is unique up to deformation.*

The assumption on the homology class of the fiber is necessary since, for example, $S^1 \times S^3$ fibers over S^2; but it only fails for non-trivial T^2-bundles without singular fibers and their blowups (see Remark 10.2.22 in [12]).

Conversely, we have the following result of Donaldson [8]:

Theorem 4 (Donaldson)**.** *Let (X, ω) be a compact symplectic 4-manifold. Then X carries a* symplectic Lefschetz pencil*, i.e. there exist a finite set $B \subset X$ and a map $f : X \setminus B \to \mathbb{CP}^1 = S^2$ such that f is modelled on $(z_1, z_2) \mapsto (z_1 : z_2)$ near each point of B, and f is a Lefschetz fibration with (noncompact) symplectic fibers outside of B.*

It follows immediately that the manifold $\hat{X}$ obtained from X by blowing up the points of B admits a Lefschetz fibration $\hat{f} : \hat{X} \to S^2$ with symplectic fibers, and can be described by its monodromy as discussed above.

Moreover, the fibration $\hat{f}$ has $n = |B|$ distinguished sections $e_1, \dots, e_n$, corresponding to the exceptional divisors of the blowups. Therefore, each fiber of $\hat{f}$ comes equipped with n marked points, and the monodromy of $\hat{f}$ lifts to the mapping class group of a genus g surface with n marked points.

The fact that the normal bundles of the sections e_i have degree -1 constrains the topology in an interesting manner. For example, if $\hat{f}$ is *relatively minimal* (i.e., if there are no reducible singular fibers with spherical components), then the existence of a section of square -1 implies that $\hat{f}$ cannot be decomposed as a non-trivial fiber sum (see e.g. [26]). Therefore, we restrict ourselves to the preimage of a large disc Δ containing all the chosen generators of $\pi_1(S^2 \setminus \mathrm{crit}(\hat{f}))$, and fix trivializations of the normal bundles to the sections e_i over Δ. Deleting a small tubular neighborhood of each exceptional section, we can now view the monodromy of $\hat{f}$ as a morphism

$$\hat{\psi} : \pi_1(\Delta \setminus \mathrm{crit}(\hat{f})) \to \mathrm{Map}_{g,n},$$

where $\mathrm{Map}_{g,n}$ is the mapping class group of a genus g surface with n boundary components.

The product of the Dehn twists $\tau_i = \hat{\psi}(\gamma_i)$ is no longer the identity element in $\mathrm{Map}_{g,n}$. Instead, since $\prod \gamma_i$ is homotopic to the boundary of the disc Δ, and since the normal bundle to e_i has degree -1, we have $\prod \tau_i = \delta$, where $\delta \in \mathrm{Map}_{g,n}$ is the *boundary twist*, i.e. the product of the positive Dehn twists $\delta_1, \dots, \delta_n$ along loops parallel to the boundary components.

With this understood, the previous discussion carries over, and under the assumption $2 - 2g - n < 0$ there is a one to one correspondence between factorizations of the boundary twist δ as a product of positive Dehn twists in $\mathrm{Map}_{g,n}$, up to Hurwitz equivalence and global conjugation, and genus g Lefschetz fibrations over S^2 equipped with n distinguished sections of square -1, up to isotopy.

Theorems 3 and 4 provide motivation to study the classification problem for Lefschetz fibrations, which by Proposition 2 is equivalent to the classification of mapping class group factorizations involving positive Dehn twists. Hence, various topological questions and conjectures about the classification of symplectic 4-manifolds can be reformulated as questions about mapping class group factorizations in Map_g and $\mathrm{Map}_{g,n}$. In the rest of this paper, we state and motivate a few instances of such questions for which an answer would greatly improve our understanding of symplectic 4-manifolds. Most of these questions are wide open and probably very hard.

Remarks. **(1)** The most natural invariants that one may associate to a factorization in Map_g or $\mathrm{Map}_{g,n}$ are the number r of Dehn twists in the factorization, and the normal subgroup of $\pi_1(\Sigma)$ generated by the vanishing cycles. These are both readily understood in terms of the topology of the total space M: namely, the Euler-Poincaré characteristic of M is equal to $4-4g+r$, and, assuming the existence of a section of the fibration, $\pi_1(M)$ is the quotient of $\pi_1(\Sigma)$ by the normal subgroup generated by the vanishing cycles (or equivalently, the quotient of $\pi_1(\Sigma)$ by the action of the subgroup $\mathrm{Im}(\psi) \subset \mathrm{Map}_g$). Similarly, from the intersection pairing between vanishing cycles in $H_1(\Sigma,\mathbb{Z})$ one can recover the intersection form on $H_2(M,\mathbb{Z})$. One invariant which might seem more promising is the number of *reducible* singular fibers, i.e. the number of vanishing cycles which are homologically trivial. However, it is of little practical value for the study of general symplectic 4-manifolds, because reducible singular fibers are a rare occurrence; in fact, the Lefschetz fibrations given by Theorem 4 can always be assumed to have no reducible fibers.

(2) Many of the questions mentioned below can also be formulated in terms of factorizations in the Artin braid group, or rather in the liftable subgroup of the braid group. Namely, viewing a genus g surface with boundary components as a simple branched cover of the disc, positive Dehn twists can be realized as lifts of positive half-twists in the braid group (at least as soon as the covering has degree at least 3, in the case of Dehn twists along nonseparating curves; or degree at least 4, if one allows reducible singular fibers). This corresponds to a realization of the symplectic 4-manifold X as a *branched cover* of $\mathbb{CP}^2$, from which the Lefschetz fibration can be recovered by considering the preimages of a pencil of lines in $\mathbb{CP}^2$. The reader is referred to [3] for a treatment of the classification of symplectic 4-manifolds from the perspective of branched covers and braid group factorizations. See also [7] for more background on Lefschetz fibrations and branched covers.

2. Towards a classification of Lefschetz fibrations?

In view of Proposition 2, perhaps the most important question to be asked about mapping class group factorizations is whether it is possible to classify them, at least partially. For example, it is a classical result of Moishezon and Livne [17] that genus 1 Lefschetz fibrations are always isotopic to holomorphic fibrations, and are classified by their number of vanishing cycles, which is always a multiple of 12 (assuming fibers to be irreducible; otherwise we also have to take into account the number of reducible fibers). In fact, all factorizations of the identity as a product of positive Dehn twists in $\mathrm{Map}_1 \simeq \mathrm{SL}(2,\mathbb{Z})$ are Hurwitz equivalent to one of the standard factorizations $(\tau_a\tau_b)^{6k} = 1$, where τ_a and τ_b are the Dehn twists along the two generators of $\pi_1(T^2) \simeq \mathbb{Z}^2$, and k is an integer.

Similarly, Siebert and Tian [24] have recently obtained a classification result for genus 2 Lefschetz fibrations without reducible singular fibers and with *transitive monodromy*, i.e. such that the composition of the monodromy morphism with the group homomorphism from Map_2 to S_6 which maps the standard generators τ_i, $1 \le i \le 5$ to the transpositions $(i, i+1)$ is surjective. Namely, these fibrations are all holomorphic, and are classified by their number of vanishing cycles, which is always a multiple of 10. In fact, all such fibrations can be obtained as fiber sums of two standard Lefschetz fibrations f_0 and f_1 with respectively 20 and 30 singular fibers, corresponding to the factorizations $(\tau_1\tau_2\tau_3\tau_4\tau_5\tau_5\tau_4\tau_3\tau_2\tau_1)^2 = 1$ and $(\tau_1\tau_2\tau_3\tau_4\tau_5)^6 = 1$ in Map_2, where $\tau_1, \dots, \tau_5$ are the standard generating Dehn twists. (At the level of mapping class group factorizations, the fiber sum operation just amounts to concatenation: starting from two factorizations $\tau_1 \dots \tau_r = 1$ and $\tilde\tau_1 \dots \tilde\tau_s = 1$, we obtain the new factorization $\tau_1 \dots \tau_r\tilde\tau_1 \dots \tilde\tau_s = 1$.)

On the other hand, for genus ≥ 3 (or even for genus 2 if one allows reducible singular fibers) things become much more complicated, and one can build examples of Lefschetz fibrations with non-Kähler total spaces. Thus it seems hopeless for the time being to expect a complete classification of mapping class group factorizations in all generality.

A more realistic goal might be to look for criteria which can be used to determine whether two given Lefschetz fibrations, described by their monodromy factorizations, are isotopic. The main issue at stake here is the algorithmic decidability of the *Hurwitz problem*, i.e. determining whether two given factorizations in Map_g or $\mathrm{Map}_{g,n}$ are equivalent up to Hurwitz moves (or more generally, Hurwitz moves and global conjugation).

Question 1. *Is the Hurwitz problem for mapping class group factorizations decidable? Are there interesting criteria which can be used to conclude that two given factorizations are equivalent, or inequivalent, up to Hurwitz moves and global conjugation?*

In broader terms, the question is whether mapping class group factorizations can be used to derive non-trivial and useful invariants of Lefschetz fibrations, or even better, of the underlying symplectic 4-manifolds.

At this point, it is worth mentioning two spectacular examples of such invariants which arise from geometric considerations (rather than purely from mapping class group theory). One is Seidel's construction of a Fukaya-type A_∞-category associated to a Lefschetz fibration [21], which seems to provide a computationally manageable approach to Lagrangian submanifolds and Fukaya categories in open 4-manifolds equipped with exact symplectic structures. The other is the enumerative invariant introduced by Donaldson and Smith, which counts embedded pseudoholomorphic curves in a symplectic 4-manifold by viewing them as sections of a "relative Hilbert scheme" associated to a Lefschetz fibration [9].

Remark. Generally speaking, it seems that the geometry of Lefschetz fibrations is very rich. An approach which has been developed extensively by Smith [27] is to choose an almost-complex structure on M which makes the fibration f pseudoholomorphic. The fibers then become Riemann surfaces (possibly nodal), and so we can view a Lefschetz fibration as a map $\phi : S^2 \to \overline{\mathcal{M}}_g$ with values in the compactified moduli space of genus g curves. The singular fibers correspond to intersections of $\phi(S^2)$ with the divisor Δ of nodal Riemann surfaces; hence, Lefschetz fibrations correspond to (isotopy classes of) smooth maps $\phi : S^2 \to \overline{\mathcal{M}}_g$ such that $\phi(S^2)$ intersects Δ *transversely* and *positively* (i.e., the local intersection number is always $+1$). See [27] for various results arising from this description. A related question, posed by Tian, asks whether one can find special geometric representatives for the maps ϕ, e.g. as trees of conformal harmonic maps, and use these to prove that every Lefschetz fibration decomposes into holomorphic "pieces". This statement is to be taken very loosely, since it is not true that every Lefschetz fibration breaks into a fiber sum of holomorphic fibrations; on the other hand, any Lefschetz fibration over a disc is isotopic to a holomorphic fibration [15].

We now return to our main discussion and adopt a more combinatorial point of view. A proposed invariant of Lefschetz fibrations which, if computable, could have rich applications, comes from the notion of *matching path*, as proposed by Donaldson and Seidel [23]. A matching path for a Lefschetz fibration f is an embedded arc η in $S^2 \setminus \mathrm{crit}(f)$, with end points in $\mathrm{crit}(f)$, such that the parallel transports along η of the vanishing cycles at the two end points are mutually homotopic loops in the fiber of f. For example, with the notations of §1, if two of the Dehn twists $(\tau_1, \dots, \tau_r)$ in the mapping class group factorization associated to the Lefschetz fibration are equal to each other, say $\tau_i = \tau_j$, then $\eta_i \cup \eta_j$ is a matching path. Up to the action of the braid group by Hurwitz moves, all matching paths arise in this way.

Question 2 (Donaldson)**.** *Is it possible to enumerate all matching paths in a Lefschetz fibration with given monodromy factorization?*

Geometrically, matching paths correspond to Lagrangian spheres in M (or, if considering Lefschetz fibrations with distinguished sections and their $\mathrm{Map}_{g,n}$-valued monodromy, in the blown down manifold X) [23].

Lefschetz fibrations often admit infinitely many matching paths, because isotopic Lagrangian spheres may be represented by different matching paths, and also because generalized Dehn twists can often be used to exhibit infinite families of non-isotopic Lagrangian spheres [22]. A possible solution is to ask instead which classes in $H_2(M,\mathbb{Z})$ (or $H_2(X,\mathbb{Z})$) can be represented by Lagrangian spheres arising from matching paths. Or, more combinatorially, one can look at matching paths up to the action of *automorphisms* of the Lefschetz fibration f ([6], §10). Namely, considering the action of the braid group B_r on tuples of Dehn twists by Hurwitz moves, an automorphism of the Lefschetz fibration is a braid $b \in B_r$ such that $b_*((\tau_1,\dots,\tau_r)) = (\gamma\tau_1\gamma^{-1},\dots,\gamma\tau_r\gamma^{-1})$ for some $\gamma \in \mathrm{Map}_g$. In other terms, the automorphism group is the *stabilizer* of the given monodromy factorization (or rather of its equivalence class up to global conjugation) with respect to the Hurwitz action of B_r. For example, if η is a matching path then it is easy to see that the half-twist supported along η is an automorphism of the fibration, which corresponds geometrically to the Dehn twist along the Lagrangian sphere associated to η.

If $b \in B_r$ is an automorphism of the fibration, then the image by b of any matching path is again a matching path; hence, automorphisms act on the set of matching paths. Thus, it may make more sense to consider the following question instead of Question 2: is it possible to find a set of generators of the automorphism group of a Lefschetz fibration with given monodromy factorization, and a collection of matching paths $\{\eta_j\}$ such that any matching path can be obtained from one of the η_j by applying an automorphism of the fibration?

Our next series of questions will be specific to factorizations in mapping class groups of surfaces with boundary, $\mathrm{Map}_{g,n}$, with $n > 0$. In this case, the sub-semigroup $\mathrm{Map}^+_{g,n} \subseteq \mathrm{Map}_{g,n}$ generated by positive Dehn twists is strictly contained in the mapping class group. Geometrically, assuming $g \geq 2$ and equipping Σ with a hyperbolic metric, we can use one of the distinguished sections of the Lefschetz fibration to lift the monodromy action to the universal cover. Looking at the induced action on the boundary at infinity (i.e., on the set of geodesic rays through a given point of the hyperbolic disc), it can be observed that positive Dehn twists always rotate the boundary in the *clockwise* direction [26]. This leads e.g. to the indecomposability result mentioned in §1, but also to various questions about the finiteness or uniqueness properties of factorizations of certain elements in $\mathrm{Map}_{g,n}$. To avoid obvious counterexamples arising from non relatively minimal fibrations, in the rest of the discussion we always make the following assumption on reducible singular fibers:

Assumption: every component of every fiber intersects at least one of the distinguished sections $e_1, \dots, e_n$.

In other terms, we only allow Dehn twists along closed curves which either are homologically nontrivial, or separate Σ into two components each containing at least one of the n boundary components. Then we may ask:

Question 3 (Smith)**.** *Is there an a priori upper bound on the length of any factorization of the boundary twist δ as a product of positive Dehn twists in $\mathrm{Map}_{g,n}$?*

Equivalently: is there an upper bound (in terms of the genus only) on the number of singular fibers of a Lefschetz fibration admitting a section of square -1? (In the opposite direction, various lower bounds have been established, see e.g. [28]). Unfortunately, it is hard to quantify the amount of rotation induced by a Dehn twist on the boundary of the hyperbolic disc, so it is not clear whether the approach in [26] can shed light on this question.

More generally, given an element $T \in \mathrm{Map}^+_{g,n}$, we can try to study factorizations of T as a product of positive Dehn twists. Geometrically, such factorizations correspond to Lefschetz fibrations over the disc (with bounded fibers), such that the monodromy along the boundary of the disc is the prescribed element T. The boundary of such a Lefschetz fibration is naturally a contact 3-manifold Y equipped with a structure of *open book* [10], and the total space of the fibration is a Stein filling of Y [1, 10, 15]. Hence the classification of factorizations of T in $\mathrm{Map}_{g,n}$ is related to (and a subset of) the classification of Stein fillings of the contact 3-manifold Y.

Some remarkable results have been obtained recently concerning the classification of symplectic fillings of lens spaces or links of singularities, using tools from symplectic geometry, and in particular pseudo-holomorphic curves (see e.g. [14, 18]); meanwhile, Lefschetz fibrations have been used to construct examples with infinitely many inequivalent fillings (see e.g. [20]). Hence we may ask:

Question 4. *For which $T \in \mathrm{Map}^+_{g,n}$ is it possible to classify factorizations of T as a product of positive Dehn twists in $\mathrm{Map}_{g,n}$? In particular, for which T is there a unique factorization, or only finitely many factorizations, up to Hurwitz equivalence and global conjugation?*

Let us now return to factorizations of the boundary twist δ, or equivalently to Lefschetz fibrations over S^2 with distinguished sections of square -1. Whereas the classification problem seems to be beyond reach, it may be a more realistic goal to search for a minimal set of moves which can be used to relate any two Lefschetz fibrations (or mapping class group factorizations) with the same genus and the same number of singular fibers to each other. At the level of 4-manifolds, this question asks for a set of surgery operations which can be used to relate any two symplectic 4-manifolds M_1 and M_2 with the same basic topological invariants to each other.

In this context, it is not necessarily useful to require the fundamental groups of the 4-manifolds M_1 and M_2 to be the same; however, it seems natural to require the Euler-Poincaré characteristics and the signatures of M_1 and M_2 to be equal to each other (in other terms, M_1 and M_2 must have the same Chern numbers c_1^2 and c_2). Moreover, when approaching this question from the angle of Lefschetz fibrations, one should require the existence of Lefschetz fibrations with the same fiber genus and with the same number of distinguished -1-sections; if considering the fibrations given by Theorem 4, this amounts to requiring the symplectic structures on M_1 and M_2 to be integral and have the same values of $[\omega]^2$ and $c_1 \cdot [\omega]$.

The constraint on Euler-Poincaré characteristics is natural, and means that we only compare Lefschetz fibrations with identical numbers of singular fibers; additionally, for simplicity it may make sense to require all singular fibers to be irreducible (as is the case for the fibrations given by Theorem 4). The signature constraint, on the other hand, is not so easy to interprete at the level of the monodromy factorizations: determining the signature from the monodromy factorization requires a non-trivial calculation, for which an algorithm has been given by Ozbagci [19] (see also [25] for a geometric interpretation).

One way in which one can try to simplify the classification of Lefschetz fibrations of a given fiber genus and with fixed Euler-Poincaré characteristic and signature is up to *stabilization* by fiber sum operations [4]. However, a more intriguing and arguably more interesting question is to understand the role played by *Luttinger surgery* in the greater topological diversity of symplectic 4-manifolds compared to complex projective surfaces (see [3] for a discussion of this problem from the viewpoint of branched covers).

Given a Lagrangian torus T in a symplectic 4-manifold, Luttinger surgery is an operation which consists of cutting out a tubular neighborhood of T, foliated by parallel Lagrangian tori, and gluing it back via a symplectomorphism wrapping the meridian around the torus (in the direction of a given closed loop on the torus), while the longitudes are not affected [16, 5]. In the context of Lefschetz fibrations, an important special case is when the torus T is fibered above an embedded loop $\gamma \subset S^2 \setminus \mathrm{crit}(f)$, with fiber an embedded closed loop α in the fiber of f (invariant under the monodromy along γ). For example, this type of Luttinger surgery accounts for the difference between twisted and untwisted fiber sums of Lefschetz fibrations (i.e., concatenating the monodromy factorizations with or without first applying a global conjugation to one of them).

Consider a Lefschetz fibration $f : M \to S^2$, a system of generating loops $\gamma_1, \dots, \gamma_r \in \pi_1(S^2 \setminus \mathrm{crit}(f))$, and integers $1 \leq k \leq l \leq r$ such that the product $\gamma_k \dots \gamma_l$ is homotopic to a given loop $\gamma \subset S^2 \setminus \Delta$. Also consider a closed loop α in the fiber, preserved by the monodromy map $\psi(\gamma_k \dots \gamma_l)$. Then we can build a torus $T \subset M$ by parallel transport of α along the loop

γ. This torus is Lagrangian for a suitable choice of symplectic structure, and Luttinger surgery along T in the direction of α amounts to a *partial conjugation* of the monodromy of f. At the level of mapping class group factorizations this corresponds to the operation

$$(\tau_1, \dots, \tau_r) \longrightarrow (\tau_1, \dots, \tau_{k-1}, t_\alpha \tau_k t_\alpha^{-1}, \dots, t_\alpha \tau_l t_\alpha^{-1}, \tau_{l+1}, \dots, \tau_r),$$

where t_α is the Dehn twist along α (and the requirement that t_α commutes with the product $\tau_k \dots \tau_l$ ensures that we obtain a valid factorization). So, we may ask the following question:

Question 5. *Given two factorizations of the boundary twist δ as a product of positive Dehn twists along nonseparating curves in $\mathrm{Map}_{g,n}$, such that the total spaces of the corresponding Lefschetz fibrations have the same Euler characteristic and signature, is it always possible to obtain one from the other by a sequence of Hurwitz moves and partial conjugations?*

This question is the analogue for mapping class group factorizations of the question which asks whether any two compact integral symplectic 4-manifolds with the same $(c_1^2, c_2, [\omega]^2, c_1 \cdot [\omega])$ are related to each other via a sequence of Luttinger surgeries.

As a closing remark, let us mention that mapping class groups can shed light on the topology of symplectic manifolds not only in dimension 4, but also, with a significant amount of extra work, in dimension 6. Namely, after blowing up a finite set of points every compact symplectic 6-manifold can be viewed as a singular fibration over $\mathbb{CP}^2$, with smooth fibers everywhere except above a a singular symplectic curve $D \subset \mathbb{CP}^2$ with cusp and node singularities [2]; the fibers above the smooth points of D are nodal. Conversely, the total space of such a singular fibration over $\mathbb{CP}^2$ can be endowed with a natural symplectic structure [11]. Therefore, while in the above discussion we have focused exclusively on mapping class group factorizations, i.e. representations of the free group $\pi_1(S^2 \setminus \{\text{points}\})$ into Map_g, it may also be worthwhile to study representations of fundamental groups of plane curve complements into mapping class groups, as a possible way to further our understanding of the topology of symplectic 6-manifolds.

Acknowledgements. The author would like to thank Paul Seidel, Ivan Smith and Gang Tian for illuminating discussions about some of the questions mentioned here. The author was partially supported by NSF grant DMS-0244844.

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