

SUTURED KHOVANOV HOMOLOGY, HOCHSCHILD HOMOLOGY, AND THE OZSVÁTH-SZABÓ SPECTRAL SEQUENCE

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ABSTRACT. In [16], Khovanov-Seidel constructed a faithful action of the $(m+1)$ -strand braid group, \mathfrak{B}_{m+1} , on the derived category of left modules over a quiver algebra, A_m . We interpret the Hochschild homology of the Khovanov-Seidel braid invariant as a direct summand of the *sutured Khovanov homology* of the annular braid closure.

1. INTRODUCTION

In [14], Khovanov constructed an invariant of links in S^3 that takes the form of a bigraded abelian group arising as the homology groups of a combinatorially-defined chain complex. The graded Euler characteristic of Khovanov’s link homology recovers the Jones polynomial.

In [1], Asaeda-Przytycki-Sikora showed how to extend Khovanov’s construction to obtain an invariant of links in any thickened surface with boundary, F . When F is an annulus, the topological situation is particularly natural; the thickened annulus, $A \times I$, can be identified with the complement of a standardly-imbedded unknot in S^3 . As observed by L. Roberts [24], the data of the imbedding $A \times I \subset S^3$ endows the Khovanov complex associated to $L \subset S^3$ with a filtration, and the resulting invariant of $L \subset (A \times I \subset S^3)$ is the filtered chain homotopy type of the complex. The induced spectral sequence converges to $\text{Kh}(L)$, the Khovanov homology of $L \subset S^3$.

This filtered complex is particularly well-suited for studying braids up to conjugacy. Explicitly, letting \mathfrak{B}_{m+1} denote the $(m+1)$ -strand braid group, we can form the *annular closure*, $\widehat{\sigma} \subset A \times I$, of any braid $\sigma \in \mathfrak{B}_{m+1}$, and the isotopy class of the resulting annular link (hence, the filtered chain homotopy type of the Khovanov complex) is an invariant of the conjugacy class of $\sigma \in \mathfrak{B}_{m+1}$. Indeed, the homology of the associated graded complex, the so-called *sutured annular Khovanov homology* of $L \subset A \times I$, detects the trivial braid for every $m \in \mathbb{Z}^{\geq 0}$, though it cannot distinguish all pairs of non-conjugate braids (cf. [4]).

The main goal of this paper is to establish a relationship between the sutured Khovanov homology of a braid closure and another “categorified” braid invariant appearing in work of Khovanov and Seidel. In [16], Khovanov-Seidel consider a family of graded associative algebras, A_m , each realized as the quotient of a path algebra by a collection of quadratic relations. To each $\sigma \in \mathfrak{B}_{m+1}$ they associate a differential graded bimodule, \mathcal{M}_σ , over A_m , well-defined up to homotopy equivalence, and prove that tensoring with \mathcal{M}_σ over A_m yields a well-defined endofunctor on $D^b(A_m\text{-mod})$, the bounded derived category of left A_m -modules. By relating the action of \mathfrak{B}_{m+1} on $D^b(A_m\text{-mod})$ to its action on the Fukaya category of a certain symplectic manifold (the Milnor fiber of an A_m -type singularity), they prove that their categorical action is *faithful*, i.e., $\mathcal{M}_\sigma \cong \mathcal{M}_{\mathbb{1}}$ iff $\sigma = \mathbb{1}$. The Khovanov-Seidel

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construction also gives rise to a braid conjugacy class invariant: the *Hochschild homology* of A_m with coefficients in \mathcal{M}_σ , denoted $HH(A_m, \mathcal{M}_\sigma)$.

Our main result is a proof that these two braid conjugacy invariants are closely-related; one is a direct summand of the other. To state our result more precisely, we first note that while Khovanov homology is *bigraded*, sutured annular Khovanov homology is *triply-graded* (the extra *filtration* grading appropriately measures “wrapping” around the annulus). Denote by $\mathrm{SKh}(L; f)$ the sutured Khovanov homology in filtration grading f . We prove:

Theorem 5.1. *Let $\sigma \in \mathfrak{B}_{m+1}$, and $m(\hat{\sigma}) \subset A \times I$ the mirror of its annular closure.*

$$\mathrm{SKh}(m(\hat{\sigma}); m-1) \cong HH(A_m, \mathcal{M}_\sigma).$$

Informally, the Hochschild homology of the Khovanov-Seidel bimodule, \mathcal{M}_σ , agrees with the “next-to-top” graded piece of $\mathrm{SKh}(m(\hat{\sigma}))$.¹ Moreover, up to an overall normalization, the bigrading on $\mathrm{SKh}(\hat{\sigma})$ agrees with a natural bigrading on $HH(A_m, \mathcal{M}_\sigma)$. This is made explicit in the more precise version of Theorem 5.1 stated in Section 5.

Those readers familiar with strongly-based mapping class bimodules in bordered Heegaard-Floer homology should recognize Theorem 5.1 as a Khovanov homology analogue of (the 1-moving strand case of) [19, Thm. 7]. Indeed, letting $\Sigma(\hat{\sigma})$ denote the double-cover of S^3 branched over $\hat{\sigma}$, $p: \Sigma(\hat{\sigma}) \rightarrow S^3$ the covering map, and K_B the braid axis, one corollary of Theorem 5.1 is a new proof of the existence of a spectral sequence relating the “next-to-top” filtration grading of the sutured Khovanov homology of a braid closure to the “next-to-bottom” Alexander grading of the knot Floer homology of the fibered link, $p^{-1}(K_B)$. See Theorem 6.1 for a more precise statement and [24] (for even index, [8]) for the original proof of this result.

We should also point out that the Khovanov-Seidel algebra is a special case ($k=1$) of a family of algebras $A^{k,n-k}$ (where $n=m+1$), defined independently by Chen-Khovanov [6] and Stroppel [26] (see also [5]), that give rise to a categorification of the $U_q(\mathfrak{sl}_2)$ Reshetikhin-Turaev invariant of tangles (cf. [7, Thm. 66]). These algebras are subquotient algebras of Khovanov’s arc algebra H^n [15] and can be identified with endomorphism algebras of projective generators of certain blocks, $\mathcal{O}_{k,n-k}$, of category \mathcal{O} . They also admit a geometric interpretation in terms of convolution algebras of 2-block Springer fibers [27]. As in the Khovanov-Seidel setting, one obtains, for each k , a categorified braid (indeed, tangle) invariant that takes the form of a (derived equivalence class of a) bimodule, \mathcal{M}_σ^k , over the algebra $A^{k,n-k}$. The following conjecture, generalizing Theorem 5.1, arose during conversations with Catharina Stroppel:

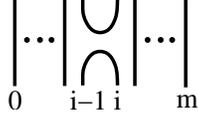
Conjecture 1.1. *Let $\sigma \in \mathfrak{B}_n$, and $m(\hat{\sigma}) \subset A \times I$ the mirror of its annular closure.*

$$\mathrm{SKh}(m(\hat{\sigma}); n-2k) \cong HH(A^{k,n-k}, \mathcal{M}_\sigma^k)$$

Establishing this conjecture may be useful for computational purposes, since modifying existing Khovanov homology programs to compute sutured Khovanov homology should be straightforward.

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¹It is an immediate consequence of the definitions (see Section 4.1) that if $\sigma \in \mathfrak{B}_{m+1}$, then $\mathrm{SKh}(\hat{\sigma}; f) = 0$ unless $f \in \{-(m+1), -(m-1), \dots, m-1, m+1\}$.

FIGURE 1. The elementary TL generator, U_i

2. TOPOLOGICAL PRELIMINARIES

As in [3, Sec. 3.1], let $D_m \subset \mathbb{C}$ denote the standard unit disk equipped with a set,

$$\Delta := \left\{ -1 + \frac{2(j+1)}{m+2} \in D_m \subset \mathbb{C} \mid j = 0, \dots, m \right\},$$

of $m+1$ points equally spaced along the real axis. $I = [0, 1]$ is the closed, positively-oriented unit interval.

Recall that the $(m+1)$ -strand braid group, \mathfrak{B}_{m+1} , is the set of equivalence classes of properly-imbedded smooth 1-manifolds $\sigma \subset D_m \times I$ satisfying the properties:

- (1) $\partial\sigma = (\Delta \times \{0\}) \cup (\Delta \times \{1\})$,
- (2) $|\sigma \cap D_m \times \{t\}| = m+1$ for all $t \in I$.

Two such 1-manifolds σ and σ' are considered *equivalent (braid isotopic)* if there exists a smooth isotopy from σ to σ' through 1-manifolds satisfying Properties (1) and (2).

\mathfrak{B}_{m+1} forms a group, with composition given by stacking (bottom to top) and vertical rescaling. We shall make frequent use of Artin's well-known presentation:

$$\mathfrak{B}_{m+1} := \left\langle \sigma_1, \dots, \sigma_m \mid \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| = 1, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i-j| \geq 2. \end{array} \right\rangle.$$

Our convention will be that Artin words read left to right denote braids formed by stacking the elementary Artin generators bottom to top.

Associated to a factorization, $\sigma = \sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}$, of $\sigma \in \mathfrak{B}_{m+1}$ as a product of elementary Artin generators, one constructs a diagram of σ in $\mathbb{R} \times I$. Any crossing of such a diagram can be resolved in one of two ways as indicated in Figure 3, and any complete resolution is an $(m+1)$ -strand *Temperley-Lieb (TL) diagram*, i.e., a smooth, properly-imbedded 1-manifold $U \subset \mathbb{R} \times I$ with $\partial U = (\Delta \times \{0\}) \cup (\Delta \times \{1\})$. As above, TL diagrams U and U' are considered equivalent if U is isotopic to U' through TL diagrams, and there is a multiplicative structure on TL diagrams, obtained by stacking and vertically rescaling. The product diagram, $\text{Id} := \Delta \times I$, is a two-sided identity with respect to this multiplication. Any TL diagram is isotopic to a product of the elementary TL diagrams U_1, \dots, U_m pictured in Figure 1, and two *TL words* in these generators represent isotopic TL diagrams iff one can be obtained from the other by applying *Jones relations*:

- (1) $U_i^2 = \bigcirc \amalg U_i$ for all $i \in \{1, \dots, m\}$
- (2) $U_i U_{i+1} U_i = U_i$ for all $i \in \{1, \dots, m-1\}$
- (3) $U_i U_{i-1} U_i = U_i$ for all $i \in \{2, \dots, m\}$

The “ \bigcirc ” of (1) represents a closed circle positioned between the cup and cap of U_i .

3. ALGEBRAIC PRELIMINARIES

We refer the reader to [12], [13], [25], [17], [18] for standard background on A_∞ algebras and modules. Terminology and notation are as in [3]. All algebras/modules we consider are over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$.

Notation 3.1. Given a bigraded vector space

$$V = \bigoplus_{i,j \in \mathbb{Z}} V_{(i,j)},$$

(e.g., a differential graded module) and $k_1, k_2 \in \mathbb{Z}$, $V[k_1]\{k_2\}$ will denote the bigraded vector space whose first (homological) grading has been shifted *down* by k_1 and whose second (internal) grading has been shifted *up* by k_2 .² Explicitly,

$$(V[k_1]\{k_2\})_{(i-k_1, j+k_2)} := V_{(i,j)}.$$

Notation 3.2. If V is a bigraded vector space, $k \in \mathbb{Z}^{\geq 0}$, and $W := \mathbb{F}_{(0,-1)} \oplus \mathbb{F}_{(0,1)}$, then we will use the notation $\bigcirc_k V$ to denote the bigraded vector space $V \otimes W^{\otimes k}$.

Notation 3.3. Let A and B be A_∞ algebras. We will denote by $D_\infty(A)$ (resp., by $D_\infty(A^{op})$, $D_\infty(A - B^{op})$) the category whose objects are homologically unital A_∞ *left-* (resp., *right-*, *bi-*) modules over A (resp., over A , over $A-B$) and whose morphisms are A_∞ homotopy classes of A_∞ morphisms.

Definition 3.4. Let A, B , and C be A_∞ algebras, $M \in D_\infty(A - B^{op})$, and $N \in D_\infty(B - C^{op})$. The derived A_∞ tensor product, $M \widetilde{\otimes}_B N$, is the well-defined object of $D_\infty(A - C^{op})$ represented by the A_∞ bimodule over $A-C$ with underlying vector space

$$\bigoplus_{k=0}^{\infty} M \otimes B^{\otimes k} \otimes N,$$

whose differential ($m_{(0|1|0)}$ structure map) is given in terms of the structure maps m^M , m^B , and m^N by

$$\begin{aligned} m_{(0|1|0)}^{M \widetilde{\otimes}_B N} &:= \sum_{i_1 \leq k} m_{(0|1|i_1)}^M \otimes \text{Id}^{\otimes k - i_1 + 1} + \sum_{i_2 \leq k} \text{Id}^{\otimes k - i_2 + 1} \otimes m_{(i_2|1|0)}^N \\ &+ \sum_{1 \leq j_1 < j_2 \leq k} \text{Id}^{\otimes j_1} \otimes m_{j_2 - j_1}^B \otimes \text{Id}^{\otimes k - j_2 + 2} \end{aligned}$$

and whose higher ($i_1 > 0$ or $i_2 > 0$) A_∞ structure maps,

$$m_{(i_1|1|i_2)}^{M \widetilde{\otimes}_B N} : A^{\otimes i_1} \otimes (M \otimes B^{\otimes k} \otimes N) \otimes C^{\otimes i_2} \rightarrow \bigoplus_{0 \leq j \leq k} M \otimes B^{\otimes k - j} \otimes N,$$

are given by

$$m_{(i_1|1|i_2)}^{M \widetilde{\otimes}_B N} := \begin{cases} \sum_{0 \leq j \leq k} m_{(i_1|1|j)}^M \otimes \text{Id}^{\otimes k - j + 1} & \text{if } i_2 = 0, \\ \sum_{0 \leq j \leq k} \text{Id}^{\otimes k - j + 1} \otimes m_{(j|1|i_2)}^M & \text{if } i_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

²The difference in shift conventions for homological versus internal gradings is unfortunate, but standard in the literature. In particular, they coincide with those in [16], to which we frequently refer.

Definition 3.5. Let A be an A_∞ algebra and $M \in D_\infty(A - A^{op})$. The *Hochschild homology* of A with coefficients in M , denoted $HH(A, M)$, is the derived A_∞ self-tensor product of M . Explicitly, it is the homology of the chain complex with underlying vector space:

$$\bigoplus_{k=0}^m A^{\otimes k} \otimes M$$

and differential

$$\begin{aligned} \partial(a_1 \otimes \dots \otimes a_k \otimes m) := & \\ \sum_{\substack{0 \leq i_1, i_2 \\ i_1 + i_2 \leq k}} a_{i_2+1} \otimes \dots \otimes a_{k-i_1} \otimes m_{(i_1|1|i_2)}^M(a_{k-i_1+1} \otimes \dots \otimes a_k \otimes m \otimes a_1 \otimes \dots \otimes a_{i_2}) & \\ + \sum_{1 \leq j_1 < j_2 \leq k} a_1 \otimes \dots \otimes m_{j_2-j_1+1}^A(a_{j_1} \otimes \dots \otimes a_{j_2}) \otimes a_{j_2+1} \otimes \dots \otimes a_k \otimes m & \end{aligned}$$

When A is an ordinary associative algebra and M is a dg bimodule, the above definition agrees with the ‘‘classical’’ definition of Hochschild homology:

Definition 3.6. Let A be an associative algebra and M a dg bimodule over A . Noting that M (resp., A) can be viewed as a left (resp., right) module over $A^e := A \otimes A^{op}$, the n th (classical) *Hochschild homology* of A with coefficients in M is

$$HH_n(A, M) := \mathrm{Tor}_n^{A^e}(A, M).$$

In particular, to compute $HH(A, M)$ one chooses a resolution

$$\mathcal{R}(A) \rightarrow A$$

of A by projective right A^e modules, and

$$HH_n(M) := H_n(\mathcal{R}(A) \otimes_{A^e} M).$$

Standard arguments in homological algebra imply that the homology is independent of the chosen resolution. When $\mathcal{R}(A)$ is the bar resolution, we obtain:

$$HH_n(A, M) := H_n(\dots \xrightarrow{\partial_4} A^{\otimes 3} \otimes M \xrightarrow{\partial_3} A^{\otimes 2} \otimes M \xrightarrow{\partial_2} A \otimes M \xrightarrow{\partial_1} M),$$

where

$$\begin{aligned} \partial_i(a_1 \otimes \dots \otimes a_i \otimes m) := & (a_1 \otimes \dots \otimes a_i \otimes \partial^M m) + \\ & (a_1 a_2 \otimes a_3 \otimes \dots \otimes a_i \otimes m) + (a_1 \otimes a_2 a_3 \otimes \dots \otimes a_i \otimes m) + \dots + (a_2 \otimes a_3 \otimes \dots \otimes m a_1), \end{aligned}$$

and ∂^M is the internal differential on M .

Since the higher structure maps of A and M are assumed to be zero, the complex described in Definition 3.5 agrees with this one. Accordingly, we shall use the definitions interchangeably when discussing the Hochschild homology of an ordinary associative algebra with coefficients in a dg bimodule.

Lemma 3.7. *Let A and B be A_∞ algebras. We have*

$$HH(A, M \widetilde{\otimes}_B N) \cong HH(B, N \widetilde{\otimes}_A M)$$

for any A_∞ A - B bimodule M and A_∞ B - A bimodule N .

Proof. The Hochschild complexes described in Definition 3.5 are chain isomorphic, via the (\mathbb{F} -linear extension of) the canonical identification

$$(a_1 \otimes \dots \otimes a_k) \otimes m \otimes (b_1 \otimes \dots \otimes b_\ell) \otimes n \longleftarrow (b_1 \otimes \dots \otimes b_\ell) \otimes n \otimes (a_1 \otimes \dots \otimes a_k) \otimes m.$$

□

Corollary 3.8. *Let A and B be A_∞ algebras. Suppose that there exists an A_∞ A - B bimodule ${}_A P_B$ and an A_∞ B - A bimodule ${}_B P_A$ satisfying the property that*

$${}_A P_B \tilde{\otimes}_B {}_B P_A \cong A \in D_\infty(A - A^{op}).$$

Then for any $M \in D_\infty(A - A^{op})$, we have

$$HH(A, M) \cong HH(B, {}_B P_A \tilde{\otimes}_A M \tilde{\otimes}_A {}_A P_B).$$

Proof. By Lemma 3.7 and the equivalence ${}_A P_B \tilde{\otimes}_B {}_B P_A \cong A$, we have

$$\begin{aligned} HH(B, {}_B P_A \tilde{\otimes}_A M \tilde{\otimes}_A {}_A P_B) &\cong HH(A, M \tilde{\otimes}_A {}_A P_B \tilde{\otimes}_B {}_B P_A) \\ &\cong HH(A, M), \end{aligned}$$

as desired. \square

We will also make use of the following non-derived version of a self-tensor product for dg bimodules over an associative algebra.

Definition 3.9. Let A be an associative algebra (i.e., a dg algebra supported in a single homological grading) and let

$$\mathcal{M} = \left(\cdots \longrightarrow \mathcal{M}_n \xrightarrow{\partial^n} \mathcal{M}_{n+1} \longrightarrow \cdots \right)$$

be a dg A -bimodule.

Then the *coinvariant quotient complex* of \mathcal{M} is the complex:

$$\mathcal{Q}(\mathcal{M}) := \left(\cdots \longrightarrow \mathcal{Q}(\mathcal{M}_n) \xrightarrow{\mathcal{Q}(\partial^n)} \mathcal{Q}(\mathcal{M}_{n+1}) \longrightarrow \cdots \right),$$

where

$$\mathcal{Q}(\mathcal{M}_n) := \frac{\mathcal{M}_n}{\langle am - ma \mid a \in A, m \in \mathcal{M}_n \rangle},$$

and $\mathcal{Q}(\partial^n)$ is the induced differential on the quotient (well-defined, since ∂^n is A -linear).

We have the following analogue of Lemma 3.7:

Lemma 3.10. *Let A, B be associative algebras, and let \mathcal{M} (resp., \mathcal{N}) be dg bimodules over $A - B$ (resp., over $B - A$). The canonical \mathbb{F} -linear map*

$$\Psi : \mathcal{Q}(\mathcal{M} \otimes_B \mathcal{N}) \rightarrow \mathcal{Q}(\mathcal{N} \otimes_A \mathcal{M})$$

sending $[m \otimes n] \in \mathcal{Q}(\mathcal{M} \otimes_A \mathcal{N})$ to $[n \otimes m] \in \mathcal{Q}(\mathcal{N} \otimes_B \mathcal{M})$ is a chain isomorphism.

Proof. Ψ is well-defined, since for any $a \in A, b \in B, m \in \mathcal{M}, n \in \mathcal{N}$ we have

$$\Psi[mb \otimes n] = [n \otimes mb] = [bn \otimes m] = \Psi[m \otimes bn]$$

and

$$\Psi[am \otimes n] = [n \otimes am] = [na \otimes m] = \Psi[m \otimes na].$$

Verifying that Ψ is a chain map (with canonical inverse) is similarly routine. \square

4. SUTURED ANNULAR KHOVANOV HOMOLOGY AND KHOVANOV-SEIDEL BIMODULES

We begin by reviewing the definition of sutured annular Khovanov homology in [1] (see also [24], [8]) as well as the construction of Khovanov-Seidel in [16].

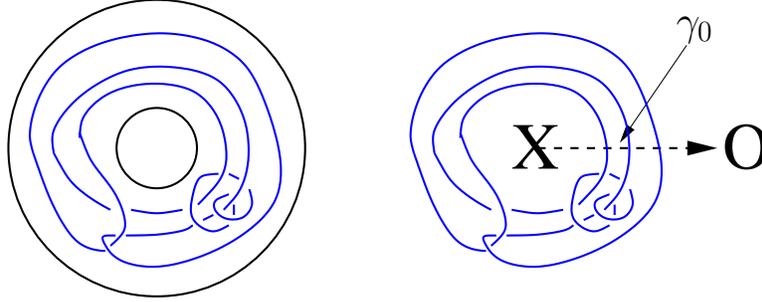


FIGURE 2. Pictured above are two diagrams of an annular link, $L \subset A \times I$, one on $A \times \{\frac{1}{2}\}$ (left) and one on S^2 equipped with two distinguished points, O and X (right). The oriented arc, γ_0 , specifies the “ f ” (filtration) grading on the Khovanov complex associated to the annular link.

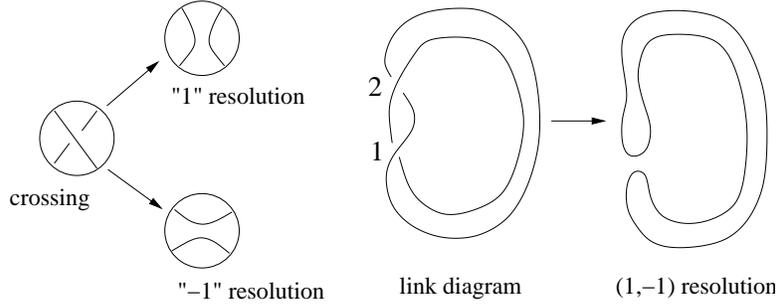


FIGURE 3. If the k crossings of a link diagram have been ordered, we can associate to each vertex, $(v_1, \dots, v_k) \in \{-1, 1\}^k$, of the k -cube a complete resolution of the diagram as indicated.

4.1. Sutured annular Khovanov homology. Sutured annular Khovanov homology is an invariant of links in a solid torus (equipped with a fixed identification as a thickened annulus), defined as follows.

Let A denote an oriented annulus and $I = [0, 1]$ the closed, positively oriented, unit interval. Any link $L \subset A \times I$ admits a diagram on $A \times \{\frac{1}{2}\}$, which can equivalently be viewed as a diagram on $S^2 - \{O, X\}$, where O, X are two distinguished points (the north and south poles, e.g.). See Figure 2.

By forgetting the data of the X basepoint, one obtains a diagram of L on $D := S^2 - N(O)$. At this point, one can construct the (bigraded) Khovanov complex as normal by ordering the k crossings of the diagram of L and assigning to the diagram its k -dimensional *cube of resolutions*, $[-1, 1]^k$, whose vertices, $\vec{v} = (v_1, \dots, v_k) \in \{-1, 1\}^k$, correspond to complete resolutions as in Figure 3.

If we now remember the data of the X basepoint, we may choose an oriented arc, γ_0 , from X to O missing all crossings of the diagram. As described in [9, Sec. 4.2], the generators of the Khovanov complex are in 1 : 1 correspondence with *enhanced* (i.e., *oriented*) resolutions, so from the data of the oriented arc we obtain an extra “ f ” (filtration) grading on the complex from the algebraic intersection number of the oriented arc with the oriented resolution. Roberts proves ([24, Lem. 1]), that the Khovanov differential is non-increasing

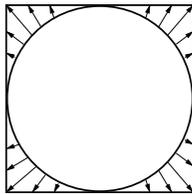


FIGURE 4. A radial slice of the homeomorphism $D_m \times S^1 \approx A \times I$

in this extra grading, so one obtains a filtration of the Khovanov complex, $\text{CKh}(L)$:

$$0 \subseteq \dots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \dots \subseteq \text{CKh}(L),$$

whose n th subcomplex is given by:

$$\mathcal{F}_n := \bigoplus_{f \leq n} \text{CKh}(L; f).$$

The sutured annular Khovanov homology of L is then defined to be the homology of the associated graded complex:

$$\text{SKh}(L) := \bigoplus_{n \in \mathbb{Z}} \text{SKh}(L; n) := \bigoplus_{n \in \mathbb{Z}} H_* \left(\frac{\mathcal{F}_n}{\mathcal{F}_{n-1}} \right).$$

A concrete description of the sutured annular chain complex is given in [24, Sec. 2]. See also [8] and [9].

In the present work, we will be interested in the special case when L is the *annular braid closure*, $\widehat{\sigma}$, of a braid, σ . Precisely, if $\sigma \subset D_m \times I$ is a braid, then we obtain $\widehat{\sigma} \subset A \times I$, by gluing $D_m \times \{0\}$ to $D_m \times \{1\}$ by the identity diffeomorphism and choosing the identification $D_m \times S^1 \approx A \times I$ that on each radial slice agrees with the homeomorphism of Figure 4.

Note that if we view S^3 as $\mathbb{R}^3 \cup \infty$, there is a standard imbedding of the annular closure of σ as a link in the complement of the braid axis,

$$K_B := \{(r, \theta, z) \mid r = 0\} \cup \infty \subset S^3,$$

via the homeomorphism identifying $A \times I$ with $S^3 - N(K_B)$:

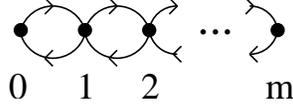
$$A \times I = \{(r, \theta, z) \mid r \in [1, 2], \theta \in [0, 2\pi), z \in [0, 1]\} \subset S^3.$$

In preparation for establishing a correspondence with the Khovanov-Seidel bimodules described in the next section, we now give an explicit description of the sutured annular chain complex of a factorized annular braid closure. Order the k crossings of $\sigma = \sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}$ from bottom to top, as read from left to right (where σ_i^{\pm} is as pictured in 6), and assign the same ordering to the crossings of the annular diagram of $\widehat{\sigma} \subset A \times I$. Endow $\widehat{\sigma}$ with the braid orientation.

Now consider the k -dimensional cube, $[-1, 1]^k$, whose vertices are indexed by k -tuples $\vec{v} = (v_1, \dots, v_k) \in \{-1, 1\}^k$. Letting $\epsilon_j(\sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}) \in \{-1, 1\}$ denote the exponent on the j th term in the product, we can now identify the vertex \vec{v} of the k -cube with the annular closure of the TL diagram, $N_{i_1} \cdots N_{i_k}$, where:

$$(4) \quad N_{i_j} := \begin{cases} U_{i_j} & \text{if } v_j \epsilon_j = 1, \text{ and} \\ \text{Id} & \text{if } v_j \epsilon_j = -1, \end{cases}$$

Moreover, if there is a directed edge from a vertex \vec{v} to a vertex \vec{w} in the k -cube then \vec{v}, \vec{w} agree except at a single entry at (say) the j -th position, where $v_j = -1$ and $w_j = +1$.

FIGURE 5. The oriented graph Γ_m used in the definition of the algebra A_m

The map assigned to this edge is one of the split/merge maps described in [24, Sec. 2], depending on the topological configuration of the splitting/merging circles.

Let n_+ (resp., n_-) denote the number of positive (resp., negative) crossings of $\hat{\sigma}$. The (h)omological, (q)uantum, and (f)iltration gradings of a sutured annular Khovanov generator, \mathbf{x} , is given in terms of its associated oriented resolution (enhanced Kauffman state), $\mathbb{S}(\mathbf{x})$, by:

$$\begin{aligned} h(\mathbf{x}) &= -n_- + \frac{1}{2} \sum_{i=1}^k (1 + v_i(\mathbf{x})) \\ q(\mathbf{x}) &= (n_+ - 2n_-) + \text{CCW}(\mathbb{S}(\mathbf{x})) - \text{CW}(\mathbb{S}(\mathbf{x})) + h(\mathbf{x}) \\ f(\mathbf{x}) &= [\gamma_0] \cdot [\mathbb{S}(\mathbf{x})], \end{aligned}$$

where “ \cdot ” denotes algebraic intersection number, and $\text{CCW}(\mathbb{S}(\mathbf{x}))$ (resp., $\text{CW}(\mathbb{S}(\mathbf{x}))$) denotes the number of components of $\mathbb{S}(\mathbf{x})$ oriented counter-clockwise (resp., clockwise).

Note that if $\sigma \in \mathfrak{B}_{m+1}$, then $\text{SKh}(\hat{\sigma}; k) = 0$ for $k \notin \{-(m+1), -(m-1), \dots, m-1, m+1\}$ (cf. [3, Prop. 7.1]), so we will sometimes refer to $\text{SKh}(\hat{\sigma}; m-1)$ as the *next-to-top* filtration level of $\text{SKh}(\hat{\sigma})$.

4.2. Khovanov-Seidel bimodules. Let Γ_m be the oriented graph (quiver) with

- vertices labeled $0, \dots, m$ and
- a pair of oppositely-oriented edges connecting each pair of adjacent vertices, as in Figure 5.

Recall that, given any oriented graph Γ , one defines its path algebra as the algebra whose underlying vector space is freely generated by the set of all finite-length paths in Γ , and multiplication is given by concatenation (the product of two non-composable paths is 0).

The algebra A_m is then defined as a quotient of the path algebra of Γ_m by the collection of relations

$$(i-1|i|i+1) = (i+1|i|i-1) = 0, \quad (i|i+1|i) = (i|i-1|i), \quad (0|1|0) = 0$$

for each $1 \leq i \leq m-1$. In the above, following [16], we have labeled each path in Γ_m by the complete ordered tuple of vertices it traverses. So, for instance, $(i|i+1|i)$ denotes the path that starts at vertex i , moves right to $i+1$, then returns to i .

The path algebra of Γ_m is further endowed with an internal grading by *negative* path length. As the above relations are homogenous with respect to this grading, it descends to the quotient, A_m .³

The collection, $\{(i|i) \mid i \in 0, \dots, m\}$, of constant paths are mutually orthogonal idempotents whose sum, $1 = \sum_{i=0}^m (i|i)$, is the identity in A_m . There are corresponding decompositions

³The reader should be warned that we are using a different internal grading than the one considered in [16] and [3]. It is this *negative path length* grading and not Khovanov-Seidel’s *steps-to-the-left* grading that is most directly related to Khovanov homology. Note also that A_m is Koszul with respect to the *positive* path length grading, cf. [26].

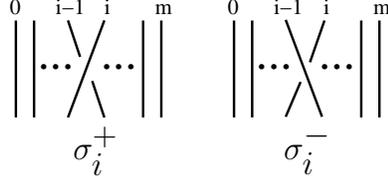


FIGURE 6. The elementary Artin braid group generators

of A_m as a direct sum of projective left-modules $A_m = \bigoplus_{i=0}^m A_m(i)$ (resp., projective right-modules $A_m = \bigoplus_{i=0}^m (i)A_m$). As in [16], we denote $A_m(i)$ (resp., $(i)A_m$) by P_i (resp., ${}_iP$). Note that P_i (resp., ${}_iP$) is the set of all paths ending at i (resp., beginning at i).

Khovanov-Seidel go on to construct a braid group action on $D^b(A_m)$, the bounded derived category of left A_m -modules, by associating to each elementary braid word $\sigma_i^{\pm 1}$ (pictured in Figure 6) a differential bimodule $\mathcal{M}_{\sigma_i^{\pm}}$ and to each braid, $\sigma := \sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}$, decomposed as a product of elementary braid words, the differential bimodule

$$\mathcal{M}_{\sigma} = \mathcal{M}_{\sigma_{i_1}^{\pm}} \otimes_{A_m} \cdots \otimes_{A_m} \mathcal{M}_{\sigma_{i_k}^{\pm}}.$$

Specifically, they associate to σ_i^+ the differential bimodule

$$\mathcal{M}_{\sigma_i^+} := P_i \otimes {}_iP\{1\} \xrightarrow{\beta_i} A_m\{1\},$$

where β_i is the A_m -bimodule map induced by the assignment $\beta_i((i) \otimes (i)) = (i)$, and to σ_i^- the differential bimodule

$$\mathcal{M}_{\sigma_i^-} := A_m \xrightarrow{\gamma_i} P_i \otimes {}_iP\{2\},$$

where

$$\gamma_i(1) = (i-1|i) \otimes (i|i-1) + (i+1|i) \otimes (i|i+1) + (i) \otimes (i|i-1|i) + (i|i-1|i) \otimes (i).$$

Suppose now that we begin with $\sigma \in \mathfrak{B}_{m+1}$, along with a decomposition $\sigma = \sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}$. Noting that the Khovanov-Seidel braid module \mathcal{M}_{σ} is a tensor product (over A_m) of two-term complexes, each of which is a mapping cone of an A_m -module map between the bimodule $(P_i \otimes {}_iP)$ assigned to the i -th elementary Temperley-Lieb (TL) object and the bimodule (A_m) assigned to the trivial TL object, we have a ‘‘cube of resolutions’’ description of \mathcal{M}_{σ} in terms of the given decomposition of σ as follows which is similar in structure to the cube of resolutions description of the sutured annular Khovanov homology of $\widehat{\sigma} \subset A \times I$.

Order the k crossings of the braid diagram $\sigma = \sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}$ in the direction of the braid orientation (by convention, from bottom to top as read from left to right). Now consider the k -dimensional cube, $[-1, 1]^k$, whose vertices are indexed by k -tuples $\vec{v} = (v_1, \dots, v_k) \in \{-1, 1\}^k$. Letting $\epsilon_j(\sigma_{i_1}^{\pm} \cdots \sigma_{i_k}^{\pm}) \in \{-1, 1\}$ denote the exponent on the j th term in the product, we can now (modulo bigrading shifts, specified in Lemma 4.1) identify the vertex \vec{v} of the k -cube with the term, $\mathcal{N}_{i_1} \otimes_{A_m} \cdots \otimes_{A_m} \mathcal{N}_{i_k}$, of the complex \mathcal{M}_{σ} , where:

$$(5) \quad \mathcal{N}_{i_j} := \begin{cases} A_m & \text{if } v_j \epsilon_j = 1, \text{ and} \\ P_{i_j} \otimes {}_{i_j}P & \text{if } v_j \epsilon_j = -1, \end{cases}$$

Moreover, if there is a directed edge from a vertex \vec{v} to a vertex \vec{w} in the k -cube then \vec{v}, \vec{w} agree except at a single entry at (say) the j -th position. The map assigned to this edge is then

$$\text{Id} \otimes \cdots \otimes \begin{Bmatrix} \beta_{i_j} \\ \gamma_{i_j} \end{Bmatrix} \otimes \cdots \otimes \text{Id}$$

according to whether $\epsilon_j = \begin{Bmatrix} +1 \\ -1 \end{Bmatrix}$.

As the homological and quantum grading shifts of generators at vertices in this ‘‘cube of resolutions’’ complex for $\mathcal{M}_\sigma = \mathcal{M}_{\sigma_{i_1}^\pm \dots \sigma_{i_k}^\pm}$ are somewhat cumbersome, we take a moment to record them here.

Lemma 4.1. *The bigraded Khovanov-Seidel bimodule associated to the vertex $\vec{v} \in \{-1, 1\}^k$ in the above cube of resolutions is*

$$(\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k}) [-\vec{v}_h] \{\vec{v}_q\},$$

where

$$\vec{v}_h := \sum_{j=1}^k \frac{1}{2}(v_j + 1),$$

and

$$\vec{v}_q = \vec{v}_h + \sum_{j=1}^k \frac{1}{2}(1 - v_j \epsilon_j).$$

Proof. By definition, \vec{v}_h is the number of 1’s in the k -tuple $\vec{v} \in \{-1, 1\}^k$, so the first statement follows.

To compute \vec{v}_q , note that we get an overall +1 shift for every $j \in \{1, \dots, n\}$ satisfying $\epsilon_j = 1$ and an extra +2 shift for each $j \in \{1, \dots, k\}$ satisfying $\epsilon_j = -1$ and $v_j = 1$. It follows that

$$\begin{aligned} \vec{v}_q &= \sum_{j=1}^k \frac{1}{2}(\epsilon_j + 1) + \sum_{j=1}^k -\frac{1}{2}(\epsilon_j - 1)(v_j + 1) \\ &= \vec{v}_h + \sum_{j=1}^k \frac{1}{2}(1 - v_j \epsilon_j), \end{aligned}$$

as desired. \square

Note that $\sum_{j=1}^k \frac{1}{2}(1 - v_j \epsilon_j)$ is the length of the TL word in elementary TL generators corresponding to the vertex \vec{v} . Equivalently, it is the number of terms of the form $(P_{i_j} \otimes_{i_j} P)$ in the corresponding tensor product $\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k}$.

5. MAIN THEOREM

In the following,

- \mathcal{M}_σ denotes the Khovanov-Seidel bimodule associated to a braid, $\sigma = \sigma_{i_1}^\pm \dots \sigma_{i_k}^\pm \in \mathfrak{B}_{m+1}$ as described in Section 4.2,
- $\text{SKh}(m(\hat{\sigma}); m-1)$ denotes the ‘‘next-to-top’’ (f)iltration grading of the sutured Khovanov homology of $m(\hat{\sigma}) \subset A \times I$, the *mirror* (all crossings reversed) of the braid closure, $\hat{\sigma}$, in $A \times I$, as in Section 4.1.

Theorem 5.1. *Let $\sigma \in \mathfrak{B}_{m+1}$ and $m(\hat{\sigma}) \subset A \times I$ the mirror of its annular closure.*

$$\text{SKh}(m(\hat{\sigma}); m-1) \cong \text{HH}(A_m, \mathcal{M}_\sigma)[n_-] \{(m-1) + n_+ - 2n_-\}$$

as bigraded vector spaces.

Remark 5.2. The *mirror* of $\hat{\sigma}$ appears on the left-hand side of the equivalence above because of differing braid conventions in [14] and [16], cf. (4) and (5).

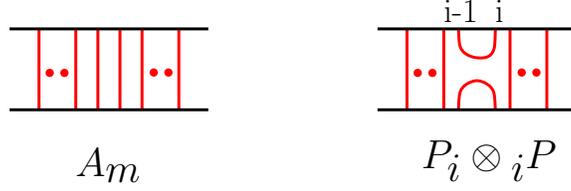


FIGURE 7. The (derived equivalence class of the) dg A_m -bimodule associated to an elementary Artin braid σ_i^\pm is a (grading-shifted) mapping cone of the bimodules A_m and $P_i \otimes_i P$; the former is identified with the trivial TL object (left) and the latter is identified with the i th elementary TL object (right).

Proof. We prove, in Proposition 5.3, that

$$\mathrm{SKh}(m(\hat{\sigma}); m-1) \cong H_*(\mathcal{Q}(\mathcal{M}_\sigma))[n_-]\{(m-1) + n_+ - 2n_-\},$$

where $H_*(\mathcal{Q}(\mathcal{M}_\sigma))$ is the homology of the coinvariant quotient module⁴ (Definition 3.9) of

$$\mathcal{M}_\sigma := \mathcal{M}_{\sigma_{i_1}^\pm \dots \sigma_{i_k}^\pm}.$$

But Proposition 5.4 tells us that

$$H_*(\mathcal{Q}(\mathcal{M}_\sigma)) \cong HH_*(A_m, \mathcal{M}_\sigma),$$

as desired. \square

Proposition 5.3. *Let $\sigma \in \mathfrak{B}_{m+1}$ be a braid of index $m+1$ and $\hat{\sigma} \subset A \times I$ its closure. Then*

$$\mathrm{SKh}(m(\hat{\sigma}); m-1) \cong H_*(\mathcal{Q}(\mathcal{M}_\sigma))[n_-]\{(m-1) + n_+ - 2n_-\}$$

as bigraded vector spaces.

Proof. As noted in Section 4.2, the Khovanov-Seidel complex \mathcal{M}_σ (and, hence, its coinvariant quotient complex $\mathcal{Q}(\mathcal{M}_\sigma)$) has a description in terms of a “cube of resolutions,” by identifying A_m with the trivial TL object and $P_i \otimes_i P$ with the i -th elementary TL object as pictured in Figure 7. The complex $\mathrm{CKh}(m(\hat{\sigma}) \subset A \times I)$ whose homology is $\mathrm{SKh}(\hat{\sigma})$ (denoted $\mathcal{C}(m(\hat{\sigma}) \subset A \times I)$ in [24, Sec. 2]—see also [1], [8, Sec. 4]) is also described in terms of a cube of resolutions. Therefore, to prove that $\mathrm{SKh}(m(\hat{\sigma}); m-1)$ coincides with $\mathcal{Q}(\mathcal{M}_\sigma)$ (up to the stated grading shift) it suffices to verify that the two cubes of resolutions assign isomorphic bigraded \mathbb{F} -vector spaces to the vertices and that, with respect to this isomorphism, the edge maps agree.

We begin with the vertices of the cube, comparing case-by-case:

- the coinvariant quotient, $\mathcal{Q}(\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k})$, of the Khovanov-Seidel bimodule associated to a vertex, and
- the \mathbb{F} -vector space associated to the corresponding vertex in the “next-to-top” grading of the sutured Khovanov complex, $\mathrm{CKh}(m(\hat{\sigma}); m-1)$.

See Figure 8 for TL diagrams associated to the various cases.

⁴It is proven in [16] that the homotopy equivalence class of \mathcal{M}_σ does not depend on the chosen factorization of σ as a product of elementary braids. Since any homotopy equivalence descends to the coinvariant quotient, we are justified in suppressing the factorization from the notation.

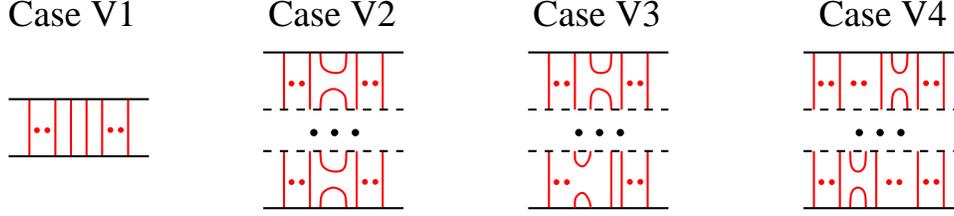


FIGURE 8. We have enumerated above the different types of TL diagrams that can appear at the vertices of a cube of resolutions.

In what follows, we shall always assume that we have used the isomorphism

$$M \otimes_{A_m} A_m \cong A_m \otimes_{A_m} M \cong M$$

to eliminate extraneous copies of A_m in the tensor product associated to a vertex. Whenever we refer to a *trivial* (resp., *nontrivial*) circle in a resolution of the annular closure, we mean a component of the resolution that represents the trivial (resp., nontrivial) element of $H_1(A; \mathbb{Z}/2\mathbb{Z})$.

Case V1: $\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k} \cong A_m$

Khovanov-Seidel coinvariant quotient:

Note that

$$A_m = \bigoplus_{i,j=0}^m {}_i P_j$$

where ${}_i P_j := {}_i P \otimes_{A_m} P_j$. The coinvariant quotient, $\mathcal{Q}(A_m)$ is, by definition, the quotient of the (dg) bimodule A_m by the ideal generated by the image of the map $\phi : A_m \otimes A_m \rightarrow A_m$ defined by $\phi(a \otimes b) = ab + ba$.

Therefore any element of ${}_i P_j$ for $i \neq j$ becomes 0 when we pass to $\mathcal{Q}(A_m)$, since each such element can be expressed as $\phi[(i) \otimes (i \dots | j)]$. Furthermore, any element of ${}_i P_i$ of the form $(i|i-1|i) = (i|i+1|i)$ is also 0 in $\mathcal{Q}(A_m)$, since we have

$$(i|i-1|i) = (i-1|i|i-1) + \phi[(i|i-1) \otimes (i-1|i)].$$

Now, noting that $(i-1|i|i-1) = (i-1|i-2|i-1) \in A_m$ and iterating, we see that $(i|i-1|i) + (0|1|0)$ is in the image of ϕ , but $(0|1|0) = 0 \in A_m$.

The only possible nonzero elements of $\mathcal{Q}(A_m)$ are therefore the idempotents, $\{(i) \in {}_i P_i\}$. But these can only be decomposed as $(i) \otimes (i) \in A_m \otimes A_m$, so we obtain no new relations among them when we pass to the coinvariant quotient.

We conclude that the vector space associated to a vertex of this type is $(m+1)$ -dimensional, with basis given by the idempotents $(0), (1), \dots, (m)$. In other words, the Khovanov-Seidel coinvariant quotient complex assigns the (grading-shifted) bigraded vector space

$$\mathbb{F}_{(\vec{v}_h, \vec{v}_q)}^{m+1}[n_-] \{(m-1) + (n_+ - 2n_-)\}$$

to the vertex $\vec{v} = (v_1, \dots, v_k) \in \{-1, 1\}^k$.

Sutured annular Khovanov complex:

The vertex associated to the closure of the trivial TL object in the cube of resolutions for $\text{CKh}(m(\hat{\sigma}); m-1)$ is also $(m+1)$ -dimensional, since the resolution consists of $m+1$ nontrivial circles. The vector space associated to this vertex in the f grading $m-1$ (the

“next-to-top” one) has a basis given by the $m + 1$ enhanced resolutions where exactly one of the circles has been labeled v_- , and the rest have been labeled v_+ . The bigraded vector space associated to the sutured Khovanov complex at vertex \vec{v} is therefore:

$$\mathbb{F}_{(\vec{v}_h, (m-1)+\vec{v}_h)}^{m+1}[n_-]\{(n_+ - 2n_-)\}.$$

But $\vec{v}_h = \vec{v}_q$ in this case (cf. Lemma 4.1), so the two bigraded vector spaces agree.

Isomorphism:

For $i = 0, 1, \dots, m$, let θ_i denote the basis element of $\text{CKh}(m(\hat{\sigma}); m-1)$ described above whose i th circle is labeled v_- . Then the linear map $\Phi : \mathcal{Q}(A_m) \rightarrow \text{CKh}(m(\hat{\sigma}); m-1)$:

$$\Phi[(i)] = \begin{cases} \theta_i & \text{if } i = 0, \text{ and} \\ \theta_i + \theta_{i-1} & \text{otherwise} \end{cases}$$

is an isomorphism of bigraded \mathbb{F} -vector spaces (strategically chosen so that the boundary maps along the edges of the cube will agree).

Case V2: $\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k} = (P_{i_{j_1}} \otimes_{i_{j_1}} P) \otimes_{A_m} \dots \otimes_{A_m} (P_{i_{j_n}} \otimes_{i_{j_n}} P)$, where $i_{j_1} = i_{j_n}$. (We allow here the possibility that $n = 1$.)

Khovanov-Seidel:

Rewrite the above tensor product as:

$$P_{i_{j_1}} \otimes (i_{j_1} P \otimes_{A_m} P_{i_{j_2}}) \otimes \dots \otimes (i_{j_{n-1}} P \otimes_{A_m} P_{i_{j_n}}) \otimes i_{j_n} P$$

as in the proof of [16, Thm. 2.2], and note that

$${}_a P \otimes_{A_m} P_b = \begin{cases} \text{Span}_{\mathbb{F}}\{(a), (a|a-1|a)\} & \text{if } |a-b| = 0, \\ \text{Span}_{\mathbb{F}}\{(a|b)\} & \text{if } |a-b| = 1, \text{ and} \\ 0 & \text{if } |a-b| > 1. \end{cases}$$

The corresponding Khovanov-Seidel bimodule is therefore 0 if there exists $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$. Now suppose there exists no such adjacent pair. Then if the associated TL diagram has ℓ closed components, we claim that its closure will have:

- $\ell + 1$ trivial closed circles and
- $m - 1$ additional nontrivial circles.

That there are $\ell + 1$ trivial closed circles in the closure is clear. To see that there are exactly $m - 1$ nontrivial circles in the closure, proceed by induction on n , the length of the associated TL word. The base case ($n = 1$) is quickly verified. If $n > 1$, the assumptions

- (1) There exists no $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$, and
- (2) $i_{j_1} = i_{j_n}$

imply that there exists some subword of the TL word that can be replaced by a shorter subword using one of the Jones relations: (1)-(3), and the claim follows.

Corresponding applications of [16, Thm 2.2]⁵ now allow us to replace the original Khovanov-Seidel bimodule with the quasi-isomorphic bimodule $\bigcirc_{\ell} (P_{i_{j_1}} \otimes_{i_{j_1}} P)\{-(n-1)\}$.

We therefore need only understand $\mathcal{Q}(P_{i_{j_1}} \otimes_{i_{j_1}} P)$. An analysis similar to the one conducted in Case V1 implies that $\mathcal{Q}(P_{i_{j_1}} \otimes_{i_{j_1}} P)$ is free of rank 2, generated by $(i_{j_1}) \otimes (i_{j_1})$ and $(i_{j_1}) \otimes (i_{j_1}|i_{j_1}-1|i_{j_1})$ (identified with $(i_{j_1}-1|i_{j_1}) \otimes (i_{j_1}|i_{j_1}-1)$ and $(i_{j_1}|i_{j_1}-1|i_{j_1}) \otimes (i_{j_1})$ in the coinvariant quotient module).

⁵Note that, since we are using the *negative path length* rather than *steps-to-the-left* grading, we have $\{-2\}$ rather than $\{1\}$ shifts on the RHS of [16, Eqns. 2.2-2.4].

We conclude that the (grading-shifted) Khovanov-Seidel coinvariant quotient associated to a vertex of this type is 0 if there exists $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$, and

$$\bigcirc_{\ell+1} \mathbb{F}_{(\vec{v}_h, \vec{v}_q - n)}[n_-] \{(m-1) + (n_+ - 2n_-)\}$$

otherwise.

Sutured Khovanov:

Suppose that there exists $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$. Then the closure of the corresponding TL diagram can have no more than $m-3$ nontrivial circles. The vector space associated to a vertex of this type in this case is therefore 0, as it is in the Khovanov-Seidel setting.

Now suppose there exists no such adjacent pair. As explained above, if the associated TL diagram has ℓ closed components, its closure will have $\ell+1$ trivial closed circles and $m-1$ additional nontrivial circles.

The $\text{CKh}(m(\hat{\sigma}); m-1)$ vector space therefore has a basis in 1 : 1 correspondence with enhanced resolutions whose $(m-1)$ nontrivial circles have all been labeled v_+ and the $(\ell+1)$ trivial circles have been labeled with either w_{\pm} . The associated (grading-shifted) bigraded vector space is therefore:

$$\bigcirc_{\ell+1} \mathbb{F}_{(\vec{v}_h, (m-1)+\vec{v}_h)}[n_-] \{n_+ - 2n_-\}.$$

Since $n = \sum_{j=1}^k (1 - v_j \epsilon_j)$ (cf. Lemma 4.1), the two bigraded vector spaces agree.

Isomorphism:

If there exists $a \in \{1, \dots, n-1\}$ with $|i_{j_a} - i_{j_{a+1}}| > 1$, the vector spaces (and isomorphism) are trivial.

Otherwise, each of the $(\ell+1)$ trivial circles in the closure of the TL diagram corresponds to either an adjacent pair

$$\dots \otimes (i_{j_a} P \otimes_{A_m} P_{i_{j_{a+1}}}) \otimes \dots$$

with $i_{j_a} = i_{j_{a+1}}$ or to the pair of outer terms

$$P_{i_{j_1}} \otimes \dots \otimes i_{j_n} P,$$

which by assumption also satisfy $i_{j_1} = i_{j_n}$.

In fact, the basis elements of $\mathcal{Q}(\mathcal{M}_\sigma)$ are in 1 : 1 correspondence with labelings of the corresponding resolved diagram, where each trivial circle is labeled with either the length 0 path (i_{j_a}) or the length 2 path ($i_{j_a} |i_{j_a} - 1| i_{j_a}$). Similarly, the basis elements of $\text{CKh}(m(\hat{\sigma}); m-1)$ are in 1 : 1 correspondence with labelings of the resolved diagram, where each trivial circle is labeled with either a w_+ or a w_- .

We therefore obtain an isomorphism

$$\Phi : \mathcal{Q}((P_{i_{j_1}} \otimes_{i_{j_1}} P) \otimes_{A_m} \dots \otimes_{A_m} (P_{i_{j_n}} \otimes_{i_{j_n}} P)) \rightarrow \text{CKh}(\hat{\sigma}; m-1)$$

by identifying the length 0 (resp., length 2) path labels on the Khovanov-Seidel side with the w_+ (resp., w_-) labels on the sutured Khovanov side.

Case V3: $\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k} = (P_{i_{j_1}} \otimes_{i_{j_1}} P) \otimes_{A_m} \dots \otimes_{A_m} (P_{i_{j_n}} \otimes_{i_{j_n}} P)$, where $|i_{j_1} - i_{j_n}| = 1$.

Khovanov-Seidel:

As in Case V2, the vertex is assigned 0 if there exists $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$. If there does not exist such an adjacent pair, then the analysis proceeds much as in Case V2. If the TL diagram has ℓ closed components, then its closure will have:

- ℓ trivial circles and
- $(m-1)$ nontrivial circles,

by an inductive argument analogous to the one used in Case V2. The corresponding Khovanov-Seidel bimodule is isomorphic to $\langle \ell \rangle (P_{i_{j_1}} \otimes_{i_{j_n}} P) \{-(n-1)\}$.

Since $|i_{j_1} - i_{j_n}| = 1$, $\mathcal{Q}(P_{i_{j_1}} \otimes_{i_{j_n}} P)$ is 1-dimensional, generated by $(i_{j_1}) \otimes (i_{j_n} | i_{j_1})$.

We conclude that the (grading-shifted) Khovanov-Seidel coinvariant quotient associated to a vertex \vec{v} of this type is 0 if there exists $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$, and

$$\langle \ell \rangle \mathbb{F}_{(\vec{v}_h, \vec{v}_q - n)}[n_-] \{(m-1) + (n_+ - 2n_-)\}$$

otherwise.

Sutured Khovanov:

As before, the vector space is 0 if there exists $a \in \{1, \dots, n-1\}$ such that $|i_{j_a} - i_{j_{a+1}}| > 1$.

Otherwise, suppose that the associated TL diagram contains ℓ closed circles. Then, as above, its closure will have ℓ trivial circles and $(m-1)$ nontrivial circles. One of these $(m-1)$ nontrivial circles is distinguished by the property that it contains both the *cap* of the first elementary TL element and the *cup* of the last elementary TL element in the tensor product.

Just as in Case V2, we therefore assign the bigraded vector space:

$$\langle \ell \rangle \mathbb{F}_{(\vec{v}_h, (m-1) + \vec{v}_h)}[n_-] \{n_+ - 2n_-\}$$

and define a completely analogous isomorphism Φ .

Case V4: $\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k} = (P_{i_{j_1}} \otimes_{i_{j_1}} P) \otimes_{A_m} \dots \otimes_{A_m} (P_{i_{j_n}} \otimes_{i_{j_n}} P)$, where $|i_{j_1} - i_{j_n}| > 1$

Here we see immediately that the coinvariant quotient of the Khovanov-Seidel bimodule vanishes because there are no nonzero paths in A_m between vertices separated by distance more than 1, and the sutured annular Khovanov complex in f grading $m-1$ vanishes since the closure of the corresponding TL object can have no more than $(m-3)$ nontrivial circles.

We now verify that the edge maps agree. Again, this is a check of a finite number of cases, each related to one of those pictured in Figure 9 by a finite sequence of cyclic permutations and horizontal reflections. By applying Lemma 3.10, we may assume without loss of generality that the merge or split defining the edge map occurs in the final (top) term.

On the sutured Khovanov side, each of the edge maps corresponds to a map induced by a merge/split cobordism of one of the following types:

- *Type I:* Two nontrivial circles \longleftrightarrow One trivial circle. This corresponds to merge/split maps of the form $V \otimes V \longleftrightarrow W$ in the language of [24, Sec. 2]. Example E0 is of this type.
- *Type II:* Two trivial circles \longleftrightarrow One trivial circle. This corresponds to merge/split maps of the form $W \otimes W \longleftrightarrow W$. Examples E1a and E2a are of this type.
- *Type III:* One trivial and one nontrivial circle \longleftrightarrow One nontrivial circle. This corresponds to merge/split maps of the form $V \otimes W \longleftrightarrow V$. Examples E1b, E2b, and E2c are of this type.

While describing Cases V2-V4, we noted that every trivial circle in the annular closure of a TL element arises as a result of a (cyclically) adjacent pair

$$\dots \otimes_i P_i \otimes \dots := \dots \otimes ({}_i P \otimes_{A_m} P_i) \otimes \dots$$

in the associated Khovanov-Seidel tensor product. Moreover, “labelings” of the trivial circles (by either the length 0 or length 2 path) on the Khovanov-Seidel side correspond, via the isomorphism Φ , to labelings of the trivial circles (by either w_+ or w_-). It can now be seen

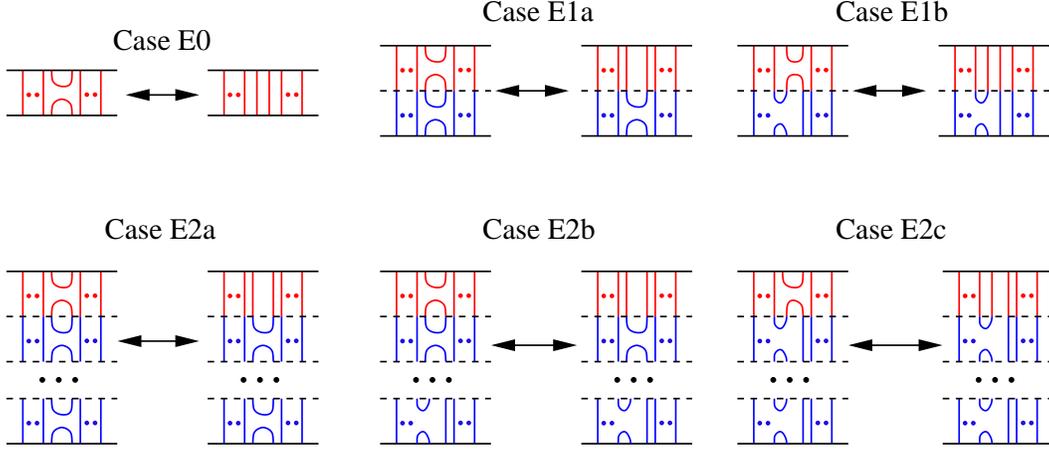


FIGURE 9. We have enumerated (modulo vertical cyclic permutation and horizontal reflection) all possible merges/splits occurring in *nontrivial* edge maps in the cube-of-resolution models for the Khovanov-Seidel and sutured Khovanov complexes. The relevant merge/split is pictured in red, and the cases are enumerated according to which TL object(s) are (cyclically) adjacent to the merge/split. Note that the edge map is necessarily 0 if the local configuration does not fall into one of the cases above, since one or both of the vertices it connects must be 0 by the argument given in Cases V2–V4.

directly that the Khovanov-Seidel maps β_i and γ_i behave exactly like the sutured Khovanov merge/split maps in all cases. We include an explicit verification of this in Cases E0, E1a, and E1b. The other cases are similar.

Case E0: $P_i \otimes_i P \longleftrightarrow A_m$

“ \rightarrow ”:

On the Khovanov-Seidel side, recall from Case V2 that $\mathcal{Q}(P_i \otimes_i P)$ has basis given by

$$\{(i) \otimes (i), (i) \otimes (i|i-1|i)\}$$

and from Case V1 that $\mathcal{Q}(A_m)$ has basis given by the idempotents

$$\{(0), \dots, (m)\}.$$

Moreover,

$$\begin{aligned} \mathcal{Q}(\beta_i)[(i) \otimes (i)] &= (i) \\ \mathcal{Q}(\beta_i)[(i) \otimes (i|i-1|i)] &= 0 \end{aligned}$$

On the sutured Khovanov side, the split map sends the generator whose trivial circle is labeled w_+ (see Case V2) to $\theta_i + \theta_{i-1}$ (see Case V1). Under the isomorphism Φ described in Cases V1 and V2, these maps agree.

“ \leftarrow ”:

The only basis elements on the Khovanov-Seidel side with nontrivial image under $\mathcal{Q}(\gamma_i)$ are the idempotents $(i-1)$ and $(i+1)$, both sent to

$$(i-1|i) \otimes (i|i-1) = (i+1|i) \otimes (i|i+1) = (i) \otimes (i|i-1|i) \in \mathcal{Q}(P_i \otimes_i P).$$

On the sutured Khovanov side, the merge map sends the generators θ_{i-1} and θ_i to the generator whose single trivial circle has been labeled w_- and whose nontrivial circles have all been labeled v_+ . Again, these maps agree under the isomorphism Φ .

Case E1a: $((P_i \otimes_i P) \otimes_{A_m} (P_i \otimes_i P)) \longleftrightarrow (P_i \otimes_i P)$

“ \rightarrow ”:

On the Khovanov-Seidel side, let ${}_i P_i$ denote $P_i \otimes_{A_m} {}_i P$ and recall from Case V2 that $\mathcal{Q}(P_i \otimes_i P_i \otimes_i P)$ has basis given by $\{(i) \otimes a \otimes b \mid a, b \in \{(i), (i|i-1|i)\}\}$, and the map $\mathcal{Q}(\beta_i)$ is given by multiplication of the last two factors:

$$\begin{aligned} \mathcal{Q}(\beta_i)[(i) \otimes (i) \otimes (i)] &= (i) \otimes (i) \\ \mathcal{Q}(\beta_i)[(i) \otimes (i|i-1|i) \otimes (i)] &= \mathcal{Q}(\beta_i)[(i) \otimes (i) \otimes (i|i-1|i)] = (i) \otimes (i|i-1|i) \\ \mathcal{Q}(\beta_i)[(i) \otimes (i|i-1|i) \otimes (i|i-1|i)] &= 0. \end{aligned}$$

On the sutured Khovanov side, the map is given by multiplying the labels w_{\pm} on the two trivial circles, which merge to form one. Now note that the isomorphism Φ identifies a generator on the Khovanov-Seidel side whose second or third tensor factor is labeled (i) (resp., $(i|i-1|i)$) with a generator on the sutured Khovanov side whose first or second trivial circle is labeled w_+ (resp., w_-). Moreover, the multiplication (merge) map in both settings is the multiplication in $\mathbb{F}[x]/x^2$ via the identification of $(i) \leftrightarrow w_+$ with 1 and $(i|i-1|i) \leftrightarrow w_-$ with x in $\mathbb{F}[x]/x^2$. The Khovanov-Seidel and sutured Khovanov maps therefore agree.

“ \leftarrow ”:

On the Khovanov-Seidel side, the map $\mathcal{Q}(\gamma_i)$ is given by:

$$\begin{aligned} \mathcal{Q}(\gamma_i)[(i) \otimes (i)] &= (i) \otimes (i) \otimes (i|i-1|i) + (i) \otimes (i|i-1|i) \otimes (i) \\ \mathcal{Q}(\gamma_i)[(i) \otimes (i|i-1|i)] &= (i) \otimes (i|i-1|i) \otimes (i|i-1|i) \end{aligned}$$

which agrees with the split map on the sutured Khovanov side under the correspondence Φ , as described above.

Case E1b: $((P_{i-1} \otimes_{i-1} P) \otimes_{A_m} (P_i \otimes_i P)) \longleftrightarrow (P_{i-1} \otimes_{i-1} P)$

“ \rightarrow ”:

On the Khovanov-Seidel side, again let ${}_{i-1} P_i$ denote ${}_{i-1} P \otimes_{A_m} P_i$ and recall from Case V3 that $\mathcal{Q}(P_{i-1} \otimes_{i-1} P_i \otimes_i P)$ is generated by $(i-1) \otimes (i-1|i) \otimes (i|i-1)$, and

$$\mathcal{Q}(\beta_i)[(i-1) \otimes (i-1|i) \otimes (i|i-1)] = (i-1) \otimes (i-1|i-2|i-1).$$

On the sutured Khovanov side, we have a single nontrivial circle splitting into one trivial and one nontrivial circle, and the split map sends the generator v_+ to the generator $v_+ \otimes w_-$. Under the isomorphism Φ from Cases V1 and V2, the two maps therefore agree.

“ \leftarrow ”:

On the Khovanov-Seidel side, we have

$$\begin{aligned} \mathcal{Q}(\gamma_i)[(i-1) \otimes (i-1)] &= (i-1) \otimes (i-1|i) \otimes (i|i-1) \\ \mathcal{Q}(\gamma_i)[(i-1) \otimes (i-1|i-2|i-1)] &= 0. \end{aligned}$$

This agrees, under the isomorphism Φ , with the sutured Khovanov merge map, which sends the generator $w_+ \otimes v_+$ to the generator v_+ and the generator $w_- \otimes v_+$ to 0. \square

Proposition 5.4. *Let $\sigma = \sigma_{i_1}^\pm \cdots \sigma_{i_k}^\pm$ be a braid. Then*

$$HH_*(A_m, \mathcal{M}_\sigma) \cong H_*(\mathcal{Q}(\mathcal{M}_\sigma)).$$

Proof. Given any resolution

$$\mathcal{R}(A_m) \xrightarrow{\partial} A_m := (\cdots R_2 \xrightarrow{\partial_2} R_1 \xrightarrow{\partial_1} R_0) \xrightarrow{\partial} A_m$$

of A_m by projective right $A_m^e := A_m \otimes A_m^{op}$ modules, $HH(A_m, \mathcal{M}_\sigma)$ is, by Definition 3.6, the homology of the complex $\mathcal{R}(A_m) \otimes_{A_m^e} \mathcal{M}_\sigma$.

Note furthermore that $\mathcal{R}(A_m) \otimes_{A_m^e} \mathcal{M}_\sigma$ has the structure of a double complex, whose ‘‘horizontal’’ differentials are of the form $\partial_* \otimes_{A_m^e} \mathbb{1}$, where ∂_* is the differential in $\mathcal{R}(A_m)$ and whose ‘‘vertical’’ differentials are of the form $\mathbb{1} \otimes_{A_m^e} d_*$, where d_* is the internal differential in the complex \mathcal{M}_σ . For example, the double complex for $\mathcal{M}_{\sigma_i^+}$ looks like:

$$\begin{array}{ccccc} \cdots R_2 \otimes_{A_m^e} (P_i \otimes_i P)\{1\} & \xrightarrow{\partial_2 \otimes \mathbb{1}} & R_1 \otimes_{A_m^e} (P_i \otimes_i P)\{1\} & \xrightarrow{\partial_1 \otimes \mathbb{1}} & R_0 \otimes_{A_m^e} (P_i \otimes_i P)\{1\} \\ \downarrow \mathbb{1} \otimes \beta_i & & \downarrow \mathbb{1} \otimes \beta_i & & \downarrow \mathbb{1} \otimes \beta_i \\ \cdots R_2 \otimes_{A_m^e} A_m\{1\} & \xrightarrow{\partial_2 \otimes \mathbb{1}} & R_1 \otimes_{A_m^e} A_m\{1\} & \xrightarrow{\partial_1 \otimes \mathbb{1}} & R_0 \otimes_{A_m^e} A_m\{1\} \end{array}$$

Let $d_h := \partial_* \otimes_{A_m^e} \mathbb{1}$ (resp., $d_v := \mathbb{1} \otimes_{A_m^e} d_*$) denote the horizontal (resp., vertical) differential on the complex $\mathcal{C}_\sigma := \mathcal{R}(A_m) \otimes_{A_m^e} \mathcal{M}_\sigma$. There is a corresponding spectral sequence converging to

$$HH(A_m, \mathcal{M}_\sigma) := H_*(\mathcal{C}_\sigma, d_h + d_v)$$

whose E^2 term is

$$H_*(H_*(\mathcal{C}_\sigma, d_h), d_v).$$

Moreover, by choosing the bar resolution:

$$\mathcal{R}(A_m) \rightarrow A_m := (\cdots A_m^{\otimes 4} \longrightarrow A_m^{\otimes 3} \longrightarrow A_m^{\otimes 2}) \longrightarrow A_m$$

we see that $\mathcal{Q}(\mathcal{M}_\sigma)$ is precisely the chain complex whose underlying vector space is $H_0(\mathcal{C}_\sigma, d_h)$ and whose differential is the induced differential, d_v , on the quotient. Hence:

$$H_*(H_0(\mathcal{C}_\sigma, d_h), d_v) \cong H_*(\mathcal{Q}(\mathcal{M}_\sigma))$$

Now we claim that $H_n(\mathcal{C}_\sigma, d_h) = 0$ for all $n \neq 0$. Since the induced differential on the E^2 page must have a nonzero horizontal degree shift, the claim implies the desired result:

$$HH(A_m, \mathcal{M}_\sigma) \cong H_*(\mathcal{Q}(\mathcal{M}_\sigma)),$$

since the spectral sequence must therefore collapse at the E^2 stage.

To prove the claim, observe first that the chain complex $(\mathcal{C}_\sigma, d_h)$ splits as a direct sum of chain complexes, one for each vertex of the cube of resolutions for \mathcal{M}_σ . Accordingly, each subcomplex is of the form:

$$\mathcal{R}(A_m) \otimes_{A_m^e} (\mathcal{N}_{i_1} \otimes_{A_m} \cdots \otimes_{A_m} \mathcal{N}_{i_k}),$$

where

$$\mathcal{N}_{i_j} := \begin{cases} A_m \\ P_{i_j} \otimes_{i_j} P \end{cases}$$

depending on the vertex of the cube; hence, the homology of each such subcomplex is

$$HH(A_m, \mathcal{N}_{i_1} \otimes_{A_m} \cdots \otimes_{A_m} \mathcal{N}_{i_k}).$$

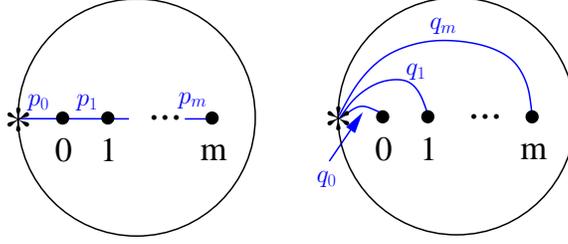


FIGURE 10. The matching and vanishing paths p_i and q_i in the base of the Lefschetz fibration $\pi : S \rightarrow \mathbb{C}$.

Now note that $P_k \otimes_{\ell} P$ is a projective A_m^e module for any $k, \ell \in \{0, \dots, m\}$, since

$$A_m \otimes A_m = \bigoplus_{k, \ell=0}^m P_k \otimes_{\ell} P.$$

Since the tensor product functor is exact on projective modules, the Hochschild homology of a projective bimodule is concentrated in degree 0. We conclude that whenever at least one of the tensor factors \mathcal{N}_{i_j} is of the form $P_{i_j} \otimes_{i_j} P$, we have

$$H_n(\mathcal{R}(A_m) \otimes_{A_m^e} (\mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k})) = 0$$

for $n \neq 0$, as desired.

The only remaining computation is $HH(A_m, \mathcal{N}_{i_1} \otimes_{A_m} \dots \otimes_{A_m} \mathcal{N}_{i_k})$ in the case $\mathcal{N}_{i_1} = \dots = \mathcal{N}_{i_k} = A_m$, i.e., $HH(A_m, A_m)$.

Rather than computing $HH(A_m, A_m)$ directly, we will exploit a relationship between A_m and another dg algebra, which we will call B_m , whose Hochschild homology is easy to compute for algebraic reasons (its homology is *directed*). The motivation here comes from the geometric interpretation of the algebras A_m and B_m in terms of the Fukaya category of a certain Lefschetz fibration, described in [2], [25, Chp. 20], and recalled briefly in [3, Sec. 3.5].

The definition of the dg algebra B_m can (and will) be given combinatorially, but symplectic geometers would do well to keep the following description in mind. Let

$$S = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 = p(z)\},$$

where p is a polynomial of degree $m+1$ whose roots are exactly the points $\mathbf{0}, \dots, \mathbf{m}$ pictured in Figure 10. Then projection to the z coordinate gives a Lefschetz fibration $\pi : S \rightarrow \mathbb{C}$, for which the arcs p_1, \dots, p_m pictured in Figure 10 are matching paths and lift to Lagrangian spheres $P_1^S, \dots, P_m^S \subset S$. Meanwhile, the vanishing paths $p_0, q_0, q_1, \dots, q_m$ determine Lefschetz thimbles $P_0^S, Q_0^S, \dots, Q_m^S$. These are respectively spherical and exceptional objects of the Fukaya category $\mathcal{F}(\pi)$ [25], in which $\text{End}(P_0^S \oplus \dots \oplus P_m^S) \simeq A_m$, whereas $\text{End}(Q_0^S \oplus \dots \oplus Q_m^S) \simeq B_m^{Kh}$. The construction carried out here is equivalent to expressing the elements of the exceptional collection Q_0^S, \dots, Q_m^S as twisted complexes built out of the generators P_0^S, \dots, P_m^S . See also [3, Sec. 3]. Note that in [3], the subscript m was dropped from the notation for B_m, B_m^{Kh} , and B_m^{HF} .

The dg algebra B_m can be described combinatorially by:

$$B_m := \bigoplus_{i, j=0}^m \text{Hom}_{A_m}(Q_i, Q_j) = \bigoplus_{i, j=0}^m i Q_j,$$

where the right/left B_m -module structure on B_m^{Kh} above is obtained in the natural way by using the quasi-isomorphism $\phi : B_m \rightarrow B_m^{Kh}$.

The proof of [3, Prop. 3.15] tells us that

$$Q \widetilde{\otimes}_{B_m} Q^* = Q \otimes_{B_m} Q^* = A_m \in D_\infty(A_m - A_m^{op}),$$

so an application of Lemma 3.7 and the observation that $\mathcal{F}^e(A_m) = B_m$ implies that $HH(A_m, A_m) \cong HH(B_m, B_m)$.

Similarly, the formality of B_m implies that

$$B_m^{Kh} \widetilde{\otimes}_{B_m^{Kh}} B_m^{Kh} = B_m^{Kh} \otimes_{B_m^{Kh}} B_m^{Kh} = B_m \in D_\infty(B_m - B_m^{op})$$

(where the first tensor factor is viewed as a $B - B^{Kh}$ bimodule and the second as a $B_m^{Kh} - B_m$ bimodule), so another application of Lemma 3.7 and the observation that $\text{Induct}_\phi(B_m) = B_m^{Kh}$ implies that $HH(B_m, B_m) \cong HH(B_m^{Kh}, B_m^{Kh})$.

It remains to compute $HH(B_m^{Kh}, B_m^{Kh})$. Since B_m^{Kh} is an associative algebra, we use the classical definition of Hochschild homology given in Definition 3.6. Accordingly, we choose the ‘‘composable’’ (projective) bar resolution of B_m^{Kh} as a left $(B_m^{Kh})^e := B_m^{Kh} \otimes (B_m^{Kh})^{op}$ module:

$$\cdots \longrightarrow \bigoplus_{i_1, \dots, i_4} (i_1 Q_{i_2}^{Kh} \otimes i_2 Q_{i_3}^{Kh} \otimes i_3 Q_{i_4}^{Kh}) \longrightarrow \bigoplus_{i_1, i_2, i_3} (i_1 Q_{i_2}^{Kh} \otimes i_2 Q_{i_3}^{Kh}),$$

which, after tensoring over $(B_m^{Kh})^e$ with B_m^{Kh} , yields the ‘‘composable’’ cyclic bar complex:

$$\cdots \xrightarrow{\partial_3} \bigoplus_{i_1, i_2, i_3} (i_1 Q_{i_2}^{Kh} \otimes i_2 Q_{i_3}^{Kh} \otimes i_3 Q_{i_1}^{Kh}) \xrightarrow{\partial_2} \bigoplus_{i_1, i_2} (i_1 Q_{i_2}^{Kh} \otimes i_2 Q_{i_1}^{Kh}) \xrightarrow{\partial_1} \bigoplus_i i Q_i$$

where, as usual,

$$\partial_k(b_1 \otimes \cdots \otimes b_{k+1}) := (b_1 b_2 \otimes \cdots \otimes b_{k+1}) + (b_1 \otimes b_2 b_3 \otimes \cdots \otimes b_{k+1}) + \cdots + (b_2 \otimes \cdots \otimes b_{k+1} b_1).$$

But, as noted previously, B_m^{Kh} is *directed*:

$$i Q_j^{Kh} := H_*(\text{Hom}_{A_m}(Q_i, Q_j)) = 0 \text{ whenever } i < j,$$

so the composable cyclic bar complex reduces to:

$$\cdots \xrightarrow{\partial_3} \bigoplus_i (i Q_i^{Kh})^{\otimes 3} \xrightarrow{\partial_2} \bigoplus_i (i Q_i^{Kh})^{\otimes 2} \xrightarrow{\partial_1} \bigoplus_i i Q_i^{Kh}.$$

Moreover, for each $i \in \{0, \dots, m\}$ and $k \in \mathbb{Z}^+$, $(i Q_i^{Kh})^{\otimes k}$ is 1-dimensional, spanned by the tensor product, $(i \mathbb{1}_i)^{\otimes k}$, of k copies of the idempotent $i \mathbb{1}_i \in i Q_i^{Kh}$. Therefore, the restricted map

$$\partial_k : (i Q_i^{Kh})^{\otimes k+1} \rightarrow (i Q_i^{Kh})^{\otimes k}$$

is given by

$$\partial_k(i \mathbb{1}_i \otimes \cdots \otimes i \mathbb{1}_i) = \begin{cases} 0 & \text{if } k \text{ is odd, and} \\ (i \mathbb{1}_i)^{\otimes k} & \text{if } k \text{ is even.} \end{cases}$$

It follows that

$$HH(B_m^{Kh}, B_m^{Kh}) \cong HH_0(B_m^{Kh}, B_m^{Kh}) \cong \bigoplus_{i=0}^m i Q_i^{Kh}$$

has dimension $m + 1$. But we calculated in Case V1 of the proof of Proposition 5.3 that $HH_0(A_m, A_m) = \mathcal{Q}(A_m)$ also has dimension $m + 1$. Since

$$m + 1 = \dim(HH_0(A_m, A_m)) \leq \dim(HH(A_m, A_m)) = \dim(HH(B_m^{Kh}, B_m^{Kh})) = m + 1,$$

it follows that $HH_n(A_m, A_m) = 0$ unless $n = 0$, as desired. \square

6. OZSVÁTH-SZABÓ SPECTRAL SEQUENCE

Theorem 5.1, when combined with [3, Thm. 6.1] and Proposition 6.3, a generalization of [19, Thm. 14], implies Theorem 6.1, a (portion of the) result originally obtained in [24] and [8] by adapting Ozsváth-Szabó's argument from [22] to Juhász's sutured Floer homology setting.

As before, $\sigma \in \mathfrak{B}_{m+1}$ denotes a braid, and $\widehat{\sigma} \subset A \times I$ denotes its annular closure (with $A \times I$ identified in the standard way with the complement of the braid axis, K_B), and $m(\widehat{\sigma})$ its mirror. In the following,

- (1) $SFH(\Sigma(\widehat{\sigma}))$ denotes the sutured Floer homology ([10]) of the double cover, $\Sigma(\widehat{\sigma})$, of the product sutured manifold $A \times I$, branched along $\widehat{\sigma} \subset A \times I$, as described in [8],
- (2) $\widehat{HFK}(\Sigma(\widehat{\sigma}), \widetilde{K}_B)$ denotes the knot Floer homology ([21],[23]) of the preimage, $\widetilde{K}_B \subset \Sigma(\widehat{\sigma})$, of K_B , and
- (3) $V = \mathbb{F}_{(\frac{1}{2}, 0)} \oplus \mathbb{F}_{(-\frac{1}{2}, -1)}$ is a “standard” 2-dimensional bigraded vector space (the subscripts on the summands indicate their (Alexander, Maslov) bigrading).

Theorem 6.1. *Let $\sigma \in \mathfrak{B}_{m+1}$. There exists a filtered chain complex whose associated graded homology is isomorphic to $SKh(m(\widehat{\sigma}); m-1)$ and whose total homology is isomorphic to the “next-to-bottom” Alexander grading of*

$$SFH(\Sigma(\widehat{\sigma})) \cong \begin{cases} \widehat{HFK}(\Sigma(\widehat{\sigma}), \widetilde{K}_B) & \text{when } m+1 \text{ is even, and} \\ \widehat{HFK}(\Sigma(\widehat{\sigma}), \widetilde{K}_B) \otimes V & \text{when } m+1 \text{ is odd,} \end{cases}$$

where \widetilde{K}_B denotes the preimage of the braid axis, K_B , in $\Sigma(\widehat{\sigma})$, the double cover of S^3 branched along $\widehat{\sigma}$.

In particular, there exists a spectral sequence from the next-to-top filtration grading of $SKh(m(\widehat{\sigma}))$ to the next-to-bottom Alexander grading of $SFH(\Sigma(\widehat{\sigma}))$.

Remark 6.2. We have chosen to match the Alexander grading normalization conventions of [10] rather than [20] in the statement of Theorem 6.1, since these fit more naturally with Zarev's conventions in [28]; we made the opposite choice in [3, Sec. 7]. Note also that in [3, Sec. 7] we asserted an isomorphism between $HH(B_m^{HF}, \mathcal{M}_\sigma^{HF})$ and the *next-to-top* (rather than *next-to-bottom*, as asserted here) Alexander grading of $\widehat{HFK}(\Sigma(\widehat{\sigma}), \widetilde{K}_B)$. The two assertions are equivalent, because of the symmetry of knot Floer homology described in [21, Sec. 3.5].

We begin by stating the following generalization of [19, Thm. 14] (replacing the pointed matched circles of [18] with the more general non-degenerate arc diagrams of [28, Defn. 2.1]), which we need for the proof of Theorem 6.1. Its proof follows quickly from Theorems 1 and 2 of Zarev's [28]:

Proposition 6.3. *Let $(Y, \Gamma, \mathcal{Z}, \phi)$ be a bordered sutured manifold in the sense of [28, Defn. 3.5], with $\mathcal{Z} = -\mathcal{Z}_1 \amalg \mathcal{Z}_2$. If there is a diffeomorphism $\Psi : G(-\mathcal{Z}_1) \rightarrow G(\mathcal{Z}_2)$ for which $\Psi|_{\mathbf{Z}_1} : \mathbf{Z}_1 \rightarrow \mathbf{Z}_2$ is orientation-preserving, then there is an identification of the Hochschild homology of the A_∞ bimodule*

$${}_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSDA}_M(Y, \Gamma)_{\mathcal{A}(\mathcal{Z}_2)}$$

(whose construction is described in [28, Sec. 8.4]) with the sutured Floer homology of (Y', Γ') , the sutured manifold obtained by self-gluing $(Y, \Gamma, \mathcal{Z}, \phi)$ along $F(-\mathcal{Z}_1) \sim_\Psi F(\mathcal{Z}_2)$ as described in [28, Sec. 3.3]:

$$SFH(Y', \Gamma') \cong HH(\mathcal{A}(\mathcal{Z}_2), {}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{BSDA}_M(Y, \Gamma)_{\mathcal{A}(\mathcal{Z}_2)}).$$

Moreover, the isomorphism appropriately identifies the “moving-strands” grading of $\mathcal{A}(\mathcal{Z}_2)$ with the “Alexander” grading of $SFH(Y', \Gamma')$ relative to $F(\mathcal{Z}_2)$.

See the end of the proof for an explicit identification of these two gradings.

Proof of Proposition 6.3. Let m denote the number of arcs in \mathcal{Z}_2 . Then $\mathcal{A}(\mathcal{Z}_2)$ splits as a direct sum:

$$\mathcal{A}(\mathcal{Z}_2) := \bigoplus_{k=0}^m \mathcal{A}(\mathcal{Z}_2, k),$$

over k -moving strands algebras as in [28, Sec. 2.2], and there is a corresponding decomposition of the bimodule:

$${}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{BSDA}_M(Y, \Gamma)_{\mathcal{A}(\mathcal{Z}_2)} := \bigoplus_{k=0}^m {}_{\mathcal{A}(\mathcal{Z}_2, k)}\widehat{BSDA}_M(Y, \Gamma, k)_{\mathcal{A}(\mathcal{Z}_2, k)}.$$

To compactify notation, we shall denote $\mathcal{A}(\mathcal{Z}_2, k)$ by \mathcal{A}_k and ${}_{\mathcal{A}(\mathcal{Z}_2, k)}\widehat{BSDA}_M(Y, \Gamma, k)_{\mathcal{A}(\mathcal{Z}_2, k)}$ by $\widehat{BSDA}_k(Y)$ for the remainder of the proof.

As alluded to in [19, Sec. 2.4.3] we have, for each k , an equivalence of categories

$$\mathcal{G} : D_\infty(\mathcal{A}_k - \mathcal{A}_k^{op}) \rightarrow D_\infty(\mathcal{A}_k \otimes \mathcal{A}_k^{op}),$$

and after replacing $\mathcal{G}(\widehat{BSDA}_k(Y))$ with a quasi-isomorphic dg model as in [19, Prop. 2.4.1], we obtain that the complex

$$CC(\mathcal{A}_k, \widehat{BSDA}_k(Y)) := \mathcal{G}(\widehat{BSDA}_k(Y)) \widetilde{\otimes}_{\mathcal{A}_k^{op} \otimes \mathcal{A}_k} \mathcal{A}_k.$$

is chain isomorphic to the Hochschild complex described in Definition 3.5.⁷ But Zarev tells us, in [28, Defn. 8.3], that $\mathcal{G}(\widehat{BSDA}_k(Y))$ is quasi-isomorphic to the (k, k) -strand module $\widehat{BSA}(Y, \Gamma, -\mathcal{Z}_1 \amalg \mathcal{Z}_2, k, k)$ (cf. [28, Sec. 1.2]), and [28, Thm. 2] tells us that the bimodule associated to

$$\text{id}_{\mathcal{Z}_2} := (F(\mathcal{Z}_2) \times I, \Lambda \times I),$$

the identity bordered sutured cobordism from $F(\mathcal{Z}_2)$ to itself (cf. [28, Defn. 1.3]), is quasi-isomorphic to the underlying algebra:

$${}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{BSDA}_M(\text{id}_{\mathcal{Z}_2})_{\mathcal{A}(\mathcal{Z}_2)} \cong \mathcal{A}(\mathcal{Z}_2) \in D_\infty(\mathcal{A}(\mathcal{Z}_2) - (\mathcal{A}(\mathcal{Z}_2))^{op}).$$

The desired result then follows from Zarev’s pairing theorem, [28, Thm. 1]. Explicitly, for each k , the total homology of

$$\begin{aligned} CC(\mathcal{A}_k, \widehat{BSDA}_k(Y)) &= \mathcal{G}(\widehat{BSDA}_k(Y)) \widetilde{\otimes}_{\mathcal{A}_k^{op} \otimes \mathcal{A}_k} \mathcal{A}_k \\ &= \widehat{BSA}(Y, \Gamma, \mathcal{Z}, k, k) \boxtimes \widehat{BSD}(\text{id}_{\mathcal{Z}_2}, k, k) \end{aligned}$$

agrees with $SFH(Y', \Gamma'; k)$, where $SFH(Y', \Gamma'; k)$ denotes the homology of the subcomplex of $CFH(Y', \Gamma')$ supported in those (relative) Spin^c structures $\mathfrak{s} \in \text{Spin}^c(Y', \Gamma')$ satisfying

$$\langle c_1(\mathfrak{s}), [F(\mathcal{Z}_2)] \rangle = \chi(F(\mathcal{Z}_2)) - |\mathbf{Z}_2| + 2k.$$

⁷Here we use the canonical correspondence between left A -modules and right A^{op} -modules.

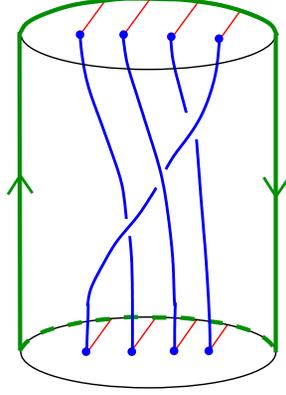


FIGURE 11. Lift the sutures (green) and parameterizing arcs (red) to obtain a bordered-sutured manifold structure on $\Sigma(\sigma)$, the double cover of $D_m \times I$ branched along σ (blue).

Here, $c_1(\mathfrak{s})$ denotes the first Chern class of \mathfrak{s} ([11, Sec. 3]) with respect to a canonical trivialization (cf. [11, Prf. of Thm. 1.5]) of v_0^\perp , and $|\mathbf{Z}_2|$ denotes the number of oriented line segments (“platforms”) of \mathcal{Z}_2 .

Informally, $SFH(Y', \Gamma'; k)$ is the “ k -from-bottom” Alexander grading of $SFH(Y', \Gamma')$ with respect to $[F(\mathcal{Z}_2)]$, the homology class of $F(\mathcal{Z}_2)$. \square

Proof of Theorem 6.1. In [3, Thm. 6.1], we

- construct a filtration on \mathcal{M}_σ^{HF} (defined in [3, Sec. 3]), the 1-moving-strand A_∞ bordered Floer bimodule over the 1-moving-strand algebra B_m^{HF} (described in [3, Rmk. 4.1]), and
- identify the associated graded complex, $\text{gr}(\mathcal{M}_\sigma^{HF})$ (an A_∞ bimodule over $B_m^{Kh} = \text{gr}(B_m^{HF})$ using [3, Thm. 5.1]) with \mathcal{M}_σ^{Kh} (defined in [3, Sec. 2] and recalled briefly in the proof of Proposition 5.4).

As in [3, Lem. 2.12], the A_∞ Hochschild complex,

$$CC(B_m^{HF}, \mathcal{M}_\sigma^{HF}) := \left(\bigoplus_{k=0}^{\infty} (B_m^{HF})^{\otimes k} \otimes \mathcal{M}_\sigma^{HF}, \partial \right),$$

inherits a filtration, and as in [3, Lem. 2.15] we see that the associated graded of the Hochschild complex is the Hochschild complex of the associated graded:

$$\text{gr}(CC(B_m^{HF}, \mathcal{M}_\sigma^{HF})) = CC(B_m^{Kh}, \mathcal{M}_\sigma^{Kh}) \in D_\infty(B_m^{Kh} - (B_m^{Kh})^{op}).$$

Now consider the bordered sutured manifold, $(\Sigma(\sigma), \Gamma, \mathcal{Z}, \phi)$, obtained from the sutured manifold $D_m \times I$ by lifting the arcs and sutures pictured in Figure 11 to a bordered sutured manifold structure on the double-cover of $D_m \times I$ branched along σ . The first author proves in [2] (cf. [3, Sec. 4]) that B_m^{HF} is isomorphic to $\mathcal{A}(\mathcal{Z}_2, 1)$ and \mathcal{M}_σ^{HF} is quasi-isomorphic to ${}_{\mathcal{A}(\mathcal{Z}_2, 1)} \widehat{BSDA}_M(\Sigma(\sigma), \Gamma, 1)_{\mathcal{A}(\mathcal{Z}_2, 1)}$. Letting $\Sigma(\hat{\sigma})$ denote the double-branched cover of the product sutured manifold $(A \times I, \Gamma')$ branched along the closure, $\hat{\sigma}$, of σ , and $\tilde{\Gamma}'$ the preimage of Γ' , Proposition 6.3 then implies:

$$SFH(\Sigma(\hat{\sigma}), \tilde{\Gamma}'; 1) \cong HH(B_m^{HF}, \mathcal{M}_\sigma^{HF}),$$

so $HH(B_m^{HF}, \mathcal{M}_\sigma^{HF})$ agrees with the next-to-bottom (\cong next-to-top) Alexander grading of

$$SFH(\Sigma(\hat{\sigma})) \cong \begin{cases} \widehat{HFK}(\Sigma(\hat{\sigma}), \tilde{K}_B) & \text{when } m+1 \text{ is even, and} \\ \widehat{HFK}(\Sigma(\hat{\sigma}), \tilde{K}_B) \otimes V & \text{when } m+1 \text{ is odd.} \end{cases}$$

See [3, Rmk. 7.1] for an explanation of the extra tensor factor of V in the odd index case.

To obtain the desired result, we now need only argue that the homology of the Hochschild complex

$$CC(B_m^{Kh}, \mathcal{M}_\sigma^{Kh}) = \text{gr}(CC(B_m^{HF}, \mathcal{M}_\sigma^{HF}))$$

agrees with $\text{SKh}(m(\hat{\sigma}); m-1)$.

But, noting that \mathcal{M}_σ^{Kh} is the image of \mathcal{M}_σ under the derived equivalences

$$D_\infty(A_m - A_m^{op}) \longleftrightarrow D_\infty(B_m - B_m^{op}) \longleftrightarrow D_\infty(B_m^{Kh} - (B_m^{Kh})^{op})$$

provided by [3, Prop. 3.15], another two applications of Lemma 3.7 as in the proof of Proposition 5.4 tells us that

$$HH(B_m^{Kh}, \mathcal{M}_\sigma^{Kh}) \cong HH(A_m, \mathcal{M}_\sigma),$$

and Theorem 5.1 then tells us that the homology of $CC(B_m^{Kh}, \mathcal{M}_\sigma^{Kh})$ is $\text{SKh}(m(\hat{\sigma}); m-1)$, as desired. \square

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