Mirror symmetry, Langlands duality and the Hitchin system

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June 2009
Workshop on mirror symmetry
MIT, Cambridge

Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; $\mathrm{G} = \mathrm{GL}_n$ or SL_n

$$\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{G}) := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \ \mathrm{G}\text{-Higgs bundles } (E,\phi) \\ \text{i.e. } E \text{ rank } n \text{ degree d bundle on } C \\ \phi \in H^0(C, ad(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{DR}}(\mathrm{G}) := \left\{ \begin{array}{c} \text{moduli space of flat } \mathrm{G}\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{B}}(\mathrm{G}) := \{A_1, B_1, .., A_g, B_g \in \mathrm{G} | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} /\!\!/ \mathrm{G}$$

- when $(d, n) = 1$ these are smooth non-compact varieties
- $\Gamma = Jac_C[n] \cong \mathbb{Z}_n^{2g}$ acts on $\mathcal{M}^d(\mathrm{SL}_n)$ by tensoring $\Rightarrow$ $\mathcal{M}^d(\mathrm{PGL}_n) := \mathcal{M}^d(\mathrm{SL}_n)/\Gamma$ is an orbifold

Theorem (Non-Abelian Hodge Theorem)

$$\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{G}) \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{DR}}(\mathrm{G}) \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{B}}(\mathrm{G})$$

Hitchin map

- the characteristic polynomial of $\phi \in H^0(C, End(E) \otimes K)$
  $\chi(\phi) \in H^0(C, K) \oplus H^0(C, K^2) \oplus \cdots \oplus H^0(C, K^n)$
  defines *Hitchin map*

$$\chi_{\mathrm{GL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{GL}_n) \to \mathbb{A}_{\mathrm{GL}_n} = \oplus_{i=1}^n H^0(C, K^i)$$

$$\chi_{\mathrm{SL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n) \to \mathbb{A}_{\mathrm{SL}_n} = \oplus_{i=2}^n H^0(C, K^i)$$

$$\chi_{\mathrm{PGL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n) \to \mathbb{A}_{\mathrm{PGL}_n} = \oplus_{i=2}^n H^0(C, K^i)$$

Theorem (Hitchin 1987)

χ is proper *and a completely integrable Hamiltonian system. Over a generic point $a \in \mathbb{A}$ the fiber $\chi^{-1}(a)$ is a torsor for an Abelian variety.*

Strominger-Yau-Zaslow

Theorem (Hausel, Thaddeus 2003)

For a generic $a \in \mathbb{A}_{\mathrm{SL}_n} \cong \mathbb{A}_{\mathrm{PGL}_n}$ *the fibers* $\chi_{\mathrm{SL}_n}^{-1}(a)$ *and* $\chi_{\mathrm{SL}_n}^{-1}(a)$ *are torsors for* dual *Abelian varieties.*

$$\begin{array}{ccc} \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n) & \leftarrow & \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n) \\ \downarrow \chi_{\mathrm{PGL}_n} & & \downarrow \chi_{\mathrm{SL}_n} \\ \mathbb{A}_{\mathrm{PGL}_n} & \cong & \mathbb{A}_{\mathrm{SL}_n}. \end{array}$$

$$\Downarrow$$

$\mathcal{M}^d_{\mathrm{DR}}(\mathrm{PGL}_n)$ *and* $\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n)$ *satisfy the SYZ construction for a pair of mirror symmetric Calabi-Yau manifolds.*

Homological Mirror Symmetry - Geometric Langlands

- (Kontsevich 1994)'s homological mirror symmetry proposal $\Rightarrow$

$$\mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n))) \sim \mathcal{D}^b(Fuk(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{PGL}_n)))$$

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- further hope $\mathcal{D}^b(Fuk(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{G})) \sim \mathcal{D}^b(\mathrm{D}mod(Bun_{\mathrm{G}}))$

  above HMS $\Rightarrow \mathcal{D}^b(\mathrm{D}mod(Bun_{\mathrm{G}})) \sim \mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{G}^{\mathrm{L}})))$

  Geometric Langlands program of (Beilinson-Drinfeld 1995) for $G = \mathrm{SL}_n$ and $G^L = \mathrm{PGL}_n$

- (Kapustin-Witten 2007) $\Rightarrow$ above from reduction of S-duality (electro-magnetic duality) in $N = 4$ SUSY YM in $4d$
- in the semi-classical limit HMS $\Rightarrow$
  $\mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n))) \cong \mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n)))$
  $\Rightarrow$ fibrewise Fourier-Mukai transform?

Topological mirror tests

- (Deligne 1972) constructs weight filtration $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H_c^k(X;\mathbb{Q})$ for any complex algebraic variety X, plus a pure Hodge structure on W_k/W_{k-1} of weight k
- define $E(X;x,y) = \sum (-1)^d x^i y^j H^{i,j}\left(W_k/W_{k-1}(H_c^d(X,\mathbb{C}))\right)$

Conjecture (Hausel–Thaddeus 2003, "DR-TMS", "Dol-TMS")

For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$, we have

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{DR}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{DR}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right).$$

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{Dol}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{Dol}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right).$$

Conjecture (Hausel–Villegas 2004,"B-TMS")

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{B}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{B}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right).$$

Results

- (Hausel–Thaddeus 2003) Dol-TMS ($\Leftrightarrow$ DR-TMS) for $n = 2, 3$ and $(d, n) = 1$ using description of $H^*(\mathcal{M}^1_{\mathrm{Dol}}(\mathrm{SL}_n))$ of (Hitchin 1987) for $n = 2$ and of (Gothen 1994) for $n = 3$
- (Hausel–Villegas $\geq$ 2004, Mereb $\geq$ 2009) B-TMS for n is prime and $n = 4$ using arithmetic techniques and character tables of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{SL}_n(\mathbb{F}_q)$
- (de Cataldo–Hausel–Migliorini, $\geq$ 2008) finds that Dol,DR and B-TMS are closely related
  $\rightsquigarrow$ surprising connection to Ngô's work on the Fundamental Lemma in the Langlands program

Weight filtration

- X variety defined over $\mathbb{Z}$
- $Frob_q : H^k(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell) \to H^k(X(\overline{\mathbb{F}}_q); \mathbb{Q}_\ell)$ Frobenius automorphism
- (Deligne 1974) proved Weil's Riemann hypothesis: eigenvalues of $Frob_q$ have absolute value $q^{i/2}$ for $i \in \mathbb{N}$
- Jordan decomposition of $Frob_q$ on $H^k \Rightarrow$ weight filtration $W_l \subset H^k$ containing all Jordan blocks of eigenvalue with modulus $q^{i/2}$ $i \leq l$
- comparison theorem: $H^*(X(\mathbb{C}); \mathbb{C}) \cong H^*(X(\overline{\mathbb{F}}_q), \mathbb{Q}_\ell) \otimes \mathbb{C}$
- (Deligne 1972) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$ for any complex algebraic variety X, which is
  - functorial
  - compatible with cup-product

Arithmetic of character variety

- For a smooth complex variety X define $E(X;q)=\sum W_i/W_{i-1}(H^k(X))(-1)^k q^{\frac{d-i}{2}}$
- (Katz 2008) proves that if $E(q):=|X(\mathbb{F}_q)|$ is polynomial in q then $E(X(\mathbb{C});q)=E(q)$
- (Hausel-Villegas 2008) calculates $E(\mathcal{M}_{\mathrm{B}};q)=|\mathcal{M}_{\mathrm{B}}(\mathbb{F}_q)|=\sum_{\chi\in Irr(\mathrm{GL}_n(\mathbb{F}_q))}\frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}}\chi(\xi_n)$
- we find $E(\mathcal{M}_{\mathrm{B}};1/q)=q^d E(\mathcal{M}_B;q)$ palindromic by *Alvis-Curtis duality*

  $q^{\frac{n(n-1)}{2}}\chi(1)(1/q)=\chi'(1)(q)$ for dual pair $\chi,\chi'\in Irr(\mathrm{GL}_n(\mathbb{F}_q))$

- $\Rightarrow$ Curious Hard Lefschetz Conjecture (theorem when $n=2$):

$$\begin{array}{rccc} L^l: & Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \rightarrow & Gr^W_{d+2l}H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ & x & \mapsto & x\cup\alpha^l \end{array},$$

where $\alpha\in W_4H^2(\mathcal{M}_{\mathrm{B}})$

Perverse filtration

- $f : X \to Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $\subset P_i \subset P_{i+1} \subset \dots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves
- recipe (de Cataldo-Migliorini, $\geq$ 2008) for perverse filtration when X smooth and Y affine:
  take $Y_0 \subset \cdots \subset Y_i \subset \dots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X)} \to \mathrm{H^k(f^{-1}(Y_i))})$$

- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{cccc} L^l : & Gr^P_{d-l}(H^*(X)) & \to & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

  where $\alpha \in H^2(X)$ is a relative ample class

Main conjecture

- recall Hitchin map $\chi: \begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} & \to & \mathbb{A} \\ (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$ is proper, thus induces perverse filtration on $H^*(\mathcal{M}_{\mathrm{Dol}})$

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism* $H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory.*

- Slogan: "Topology of the Hitchin map reflects the arithmetic of the character variety"

Theorem (de Cataldo-Hausel-Migliorini 2009)

$P = W$ *when* $G = \mathrm{GL}_2, \mathrm{PGL}_2$ *or* SL_2.

- Proof is inspired by the proof of (Ngô, 2008) of the geometrical stabilization of the trace formula

Outlook

- Define $PE(\mathcal{M}_{\mathrm{Dol}}; x, y, q) := \sum q^k E(Gr_k^P(H^*(\mathcal{M}_{\mathrm{Dol}})); x, y)$
- $PE(\mathcal{M}_{\mathrm{Dol}}; x, y, 1) = E(\mathcal{M}_{\mathrm{Dol}}; x, y) = E(\mathcal{M}_{\mathrm{DR}}; x, y)$
- Conjecture $P = W \Rightarrow PE(\mathcal{M}_{\mathrm{Dol}}; 1, 1, q) = E(\mathcal{M}_{\mathrm{B}}; q)$

Conjecture (Topological Mirror test)

$$PE_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{Dol}}^d(\mathrm{SL}_n); x, y, q\right) = PE_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{Dol}}^e(\mathrm{PGL}_n); x, y, q\right).$$

- $\Rightarrow$ Dol-TMS, DR-TMS of (Hausel-Thaddeus 2003), and B-TMS of (Hausel-Villegas 2004)
- When combined with RHL this says

$$h_{st,e}^{i,j;p}(\mathcal{M}_{\mathrm{Dol}}^d(\mathrm{SL}_n)) = h_{st,d}^{i+d_n/2-p,j+d_n/2-p;d_n-p}(\mathcal{M}_{\mathrm{Dol}}^e(\mathrm{PGL}_n))$$

  expected from fibrewise Fourier-Mukai and so from HMS
- Theorem when $n = 2$
  reflects Ngô's geometric stabilization of the trace formula
  "Fundamental Lemma for the (semi)stable nilpotent cone"?