Mark McLean – 27/5/08 – Exotic Stein manifolds

Stein mfld $\Leftrightarrow$ $i : M \hookrightarrow \mathbb{C}^N$ proper embedding
$\omega = i^*\omega_{std}$

E.g.: smooth affine varieties
∞-genus surface $z_2^2 = \sin z_1$

<u>Thm:</u> $\forall n > 3$ $\exists$ finite type pairwise non-symplectomorphic Stein mflds $(M_k)_{k \in \mathbb{N}}$ diffeomorphic to $\mathbb{R}^{2n}$

<u>NB:</u> finite type $\Rightarrow$ $\exists$ convex cylindrical end $\sim (\partial M \times [1,\infty), d(r\alpha))$

<u>Prior results:</u>
- <u>Gromov</u>: $\forall n > 1$, $\exists$ (nonstein) $(\mathbb{R}^{2n}, \omega') \not\hookrightarrow (\mathbb{R}^{2n}, \omega_{std})$
- <u>Gompf</u>: in dim. 4, $\exists$ uncountably many pairwise non-diffeo. (non-finite type) Stein manifolds homeo to $\mathbb{R}^4$
- <u>Eliashberg</u>: Any finite type Stein mfld of dim. 4 which is diffeo to $\mathbb{R}^4$ is sympl. to $(\mathbb{R}^4, \omega_{std})$
- <u>Seidel-Smith</u>: $\exists$ finite type Stein mflds diffeo to $\mathbb{R}^{4+4k}$ but $\not\hookrightarrow (\mathbb{R}^{4+4k}, \omega_{std})$

<u>Construction:</u>

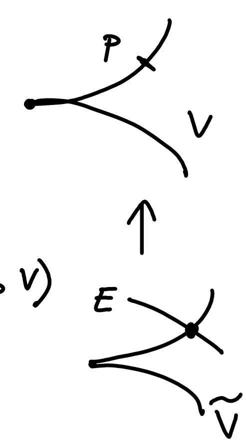
$V := \{ x^7 + y^2 + z^2 + w^2 = 0 \} \subseteq \mathbb{C}^4$

$M' := \mathbb{C}^4 \setminus V$

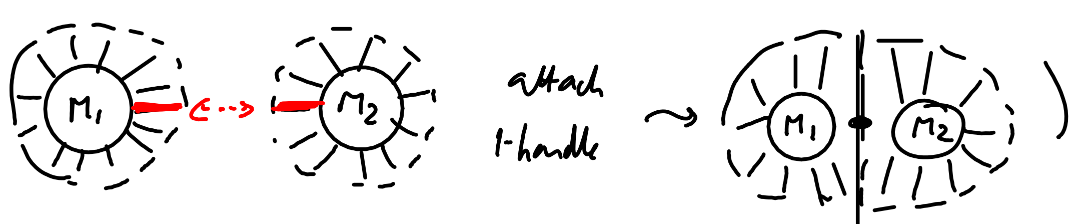
$M := \mathrm{Blowup}_p(\mathbb{C}^4) \setminus \tilde{V}$ contractible (E kills meridian to V)

$p \in V$ smooth pt — proper transform of V

$M_k := \#_{i=1}^{k} M$ end-connect sum



(Eliashberg:



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Tool to distinguish M_k's: SYMPLECTIC HOMOLOGY

$SH(N) :=$ equip N with an admissible Hamiltonian $H : N \to \mathbb{R}$
i.e. on $\partial N \times (m, \infty)$, $H = r^2$
$m >> 0$

+ an almost-complex J

$\rightsquigarrow$ $CH_*(H, J) :=$ free $\mathbb{Z}/2$-vect. space generated by 1-periodic orbits of Ham. v.f. X_H, with grading $= -i_{CZ}$.

$\partial : CH_i(H,J) \longrightarrow CH_{i-1}(H,J)$

$$\partial(o) = \sum_{-i_{CZ}(o') = -i_{CZ}(o) - 1} \#\left(\mathcal{M}(o, o')/\mathbb{R}\right) o'$$

$$\mathcal{M} = \left\{ v : \underset{s}{\mathbb{R}} \times \underset{t}{S^1} \to M \;\middle|\; \begin{array}{l} \partial_s u + J \partial_t u = \nabla H \\ u(s,t) \to o'(t) \text{ as } s \to \infty \\ o(t) \text{ as } s \to -\infty \end{array} \right\}$$

$\mathbb{R}$-action = s-translation

$SH_*(N) := H_*(CH_*(H,J))$

- Pair of pants product: $SH_i(N) \otimes SH_j(N) \longrightarrow SH_{i+j-n}(N)$

$$x \otimes y \longmapsto \sum_z \# \mathcal{M}\left(\begin{smallmatrix} x \\ y \end{smallmatrix} \; z\right) z$$

$\rightsquigarrow$ $SH_{*+n}(N)$ is a graded algebra over $\mathbb{Z}/2$ (with unit if $SH_* \neq 0$).

⚠ typically ∞-dimensional ....

- Thm. (Cieliebak) $SH_*(N_1 \#_{end} N_2) = SH_*(N_1) \oplus SH_*(N_2)$
  In particular $\mathbb{C}^n = \mathbb{C}^n \#_e \mathbb{C}^n \Rightarrow SH_*(\mathbb{C}^n) = 0$.

We get $SH_*(M_k) = \bigoplus_{i=1}^{k} SH_*(M)$

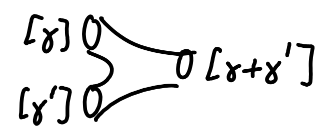
Difficulty: distinguish these when all ranks of SH_k may be ∞?

<u>Use</u>: $i(N) := \#$ idempotents of $SH_*(N)$
$(= \#\{x \,/\, x^2 = x\})$

$\to$ so $i(M_k) = i(M)^k$: enough to <u>show</u>: $1 < i(M) < \infty$

- If $SH(M) \neq 0$ then 0 and 1 are idempotents $\Rightarrow$ $i(M) \geq 2$.
- To show finiteness: look at M' (before blowup)
  $\dim_{\mathbb{C}} M' = 4 \Rightarrow SH_{4+*}(M')$ graded by $-cz-4$
  $p^2 = p \Rightarrow -cz-4 = 0$

  <u>and</u> decompose according to $H_1(M') \ni$ class of orbits
  pair-of-pants product preserves this decomp^{n}:



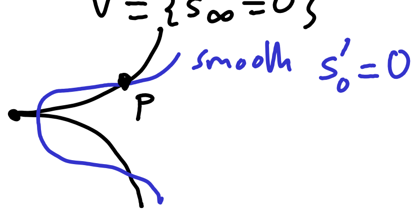
$\Rightarrow$ if $p^2 = p$ for $p = \sum \sigma_i$ then $[\sigma_i] = 0 \in H_1(M')$ $\forall i$.

<u>Claim</u>: this is enough to show $1 < i(M') < \infty$.

Next, need to relate M and M':

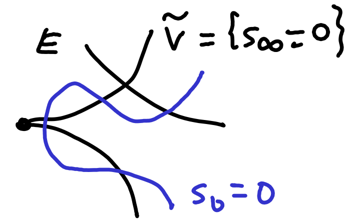
<u>Thm</u>: $SH_*(M) = SH_*(M')$.

Tool: view M, M' as Lefschetz fibrations



s'_0, s'_∞ sections of some line bundle
$\Rightarrow \pi' = s'_0 / s'_\infty : M' \to \mathbb{C}$

Then blow up at p $\rightsquigarrow$



$\to$ get $\pi = s_0 / s_\infty : M \to \mathbb{C}$

$$\begin{cases} \pi|_{M'} = \pi' \\ \pi|_{M \setminus M'} \text{ is a product} \end{cases}$$

ie. we enlarge the fibers, we don't change monodromy.

(on fiber, do a "boundary blowup" –

if $\dim_{\mathbb{C}}$ fiber $= 1$ we just fill in a puncture)

<u>Thm:</u> $p: N \to \mathbb{C}$ Lefschetz fibration (w/ convex fibers ...)

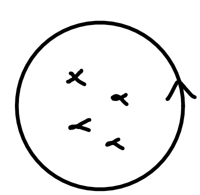
$H: \mathbb{C} \to \mathbb{R}$ admissible $\Rightarrow$ then $SH_*(N) = SH_*(p^*H, J)$

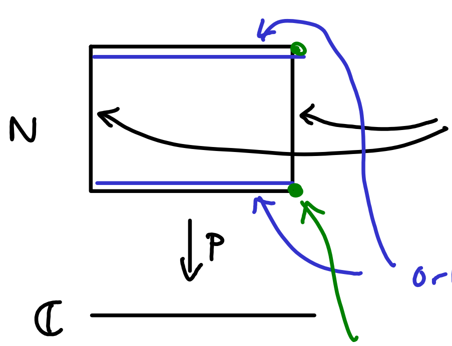
<u>ie:</u> can compute SH using a Hamiltonian pulled back from the base!

<u>Pf:</u> (sketch)

$k: N \to \mathbb{R}$ admissible $\Rightarrow CH_*(k, J) = \begin{cases} \text{Reeb orbits on } \partial N \ (\otimes H_*(S^1)) \\ H^{-*}(N) \end{cases}$

Along "vertical" boundary, look at





fixed pts of $\phi, \phi^2, \phi^3, \ldots$ give closed R. orbits

orbits are pairs (Γ, γ): Γ = Reeb orbit of ∂F

γ = generator of $CH_*(\mathbb{C})$

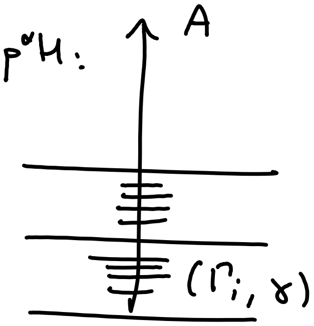
Oancea $\Rightarrow$ orbits that come from smoothing corners don't contribute

$\to CH_*(k, J) = H^{-*}(N) \oplus \mathbb{Z}/2[\text{fixed pts of } \phi, \phi^2, \phi^3, \ldots]$

$\oplus\ \mathbb{Z}/2[\text{pairs } (\Gamma, \gamma)]$

$H : \mathbb{C} \to \mathbb{R}$ admissible

$\rightsquigarrow$ action for p^*H:



Γ_i fixed
$\gamma \in CH_*(\mathbb{C})$ core clustered

but $SH_* \mathbb{C} = 0 \Rightarrow$ by spectral seq. argument these generators die early on

& we are left at some stage of spectral sequence with the smaller complex

$$H^*(N) \oplus \mathbb{Z}/2[\text{fixed pts } \phi, \phi^2, \dots] \simeq CH_*(p^*H)$$

$\blacktriangle$

(NB: • reason for using $V = \{x^7 + y^2 + z^2 + w^2 = 0\}$ in the first place is that by Brieskorn, know link is a sphere and get index gap needed for arguments.

• topologically, M = attach a 2-handle to M'

ie. fiberwise attach a 2-handle in LF's (no matter what dimension M is))