Goals. (1) Elaborate on the process of adding rays at collisions.
(2) Find an A-model (enumerative) interpretation for this process.

[Work in progress, joint with Rahul Pandharipande and Bernd Siebert.]
1. The Tropical Vertex Group (B-model)

Fix the following data:

\[ M = \mathbb{Z}^2, \quad N = \text{Hom}(M, \mathbb{Z}), \]

\[ M_\mathbb{R} = M \otimes_{\mathbb{Z}} \mathbb{R}, \quad N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R} \]
Section 1: The Tropical Vertex Group (B-model)

- \( k \) a field of characteristic zero.
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• $R$ a complete local $\mathbb{k}$-algebra, with maximal ideal $\mathfrak{m}$. 
• \( \mathbb{k} \) a field of characteristic zero.

• \( \mathbb{R} \) a complete local \( \mathbb{k} \)-algebra, with maximal ideal \( \mathfrak{m} \).

We will define a subgroup

\[
H(\mathbb{R}) \subseteq \text{Aut}(\mathbb{k}[M] \hat{\otimes}_\mathbb{k} \mathbb{R}).
\]
• \( \mathbb{k} \) a field of characteristic zero.

• \( R \) a complete local \( \mathbb{k} \)-algebra, with maximal ideal \( \mathfrak{m} \).

We will define a subgroup
\[
H(R) \subseteq \text{Aut}(\mathbb{k}[M] \hat{\otimes}_\mathbb{k} R).
\]

We will sometimes write
\[
\mathbb{k}[M] = \mathbb{k}[x^{\pm 1}, y^{\pm 1}],
\]
so an element
\[
z^m \in \mathbb{k}[M]
\]
can be written as
\[
x^a y^b
\]
if
\[
m = (a, b).
\]
**Definition.** The tropical vertex group $H(R)$ is the subgroup of $\text{Aut}(k[M] \otimes_k R)$ generated by automorphisms of the form

$$z^m \mapsto z^m f^{\langle n_0, m \rangle}$$

where

- $n_0 \in \mathbb{N}$
- $f \in k[z^{m_0}] \otimes_k R \subseteq k[M] \otimes_k R$ for some non-zero $m_0 \in M$.
- $f - 1 \in z^{m_0} m$.
- $\langle n_0, m_0 \rangle = 0$. 

Remarks.

- This is a slight variant of a group introduced by Kontsevich and Soibelman.
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- Elements of \( H(R) \) are symplectomorphisms, preserving the symplectic form

\[
\Omega = \frac{dx}{x} \wedge \frac{dy}{y}
\]

- \( z^{m_0} \) is left invariant by the automorphism

\[
z^m \mapsto z^m f^{\langle n_0, m \rangle}
\]
**Typical example.** With $R = \mathbb{k}[[t]]$,

\[
\begin{align*}
    x & \mapsto x \\
    y & \mapsto y(1 + tx)
\end{align*}
\]

is a typical example of one of the generators of $H(R)$. Here

\[
\begin{align*}
m_0 &= (1, 0) \\
n_0 &= (0, 1) \\
f &= 1 + tx
\end{align*}
\]
2. Scattering diagrams

**Definition.** A *ray* is a pair $(\mathfrak{d}, f_\mathfrak{d})$ where $\mathfrak{d} \subseteq M_\mathbb{R}$ is given by $\mathfrak{d} = m'_0 - \mathbb{R}_{\geq 0} m_0$ for some $m'_0 \in M_\mathbb{R}$ and non-zero $m_0 \in M$, and

$$f_\mathfrak{d} \in k[z^{m_0}] \hat{\otimes}_k R$$

satisfies

$$f_\mathfrak{d} - 1 \in z^{m_0} \mathfrak{m}.$$
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$$f_\mathfrak{d} \in k[z^{m_0}] \otimes_k R$$

satisfies

$$f_\mathfrak{d} - 1 \in z^{m_0}m.$$

Definition. A line is a pair $(\mathfrak{d}, f_\mathfrak{d})$ where $\mathfrak{d} \subseteq M_\mathbb{R}$ is given by $\mathfrak{d} = m'_0 - \mathbb{R}m_0$ for some $m'_0 \in M_\mathbb{R}$ and non-zero $m_0 \in M$, and

$$f_\mathfrak{d} \in k[z^{m_0}] \otimes_k R$$

satisfies

$$f_\mathfrak{d} - 1 \in z^{m_0}m.$$
Definition. A scattering diagram $\mathcal{D}$ is a set of lines and rays such that for any $n > 0$, there are only a finite number of elements $(\mathfrak{d}, f_{\mathfrak{d}})$ with

$$f_{\mathfrak{d}} \not\equiv 1 \mod m^n.$$
Consider any path
\[ \gamma : [0, 1] \rightarrow M_\mathbb{R} \]
which

- is transversal to every element of \( \mathcal{D} \) it intersects;
- does not pass through the endpoint of any ray or the intersection of any two elements;
- only passes through any given ray a finite number of times.
To such a path, we can associate a path-ordered product of automorphisms, as follows.
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First, when $\gamma$ crosses an element $(0, f_0)$, we obtain an element of $H(R)$ given by

$$z^m \mapsto z^m f_0^{\langle m, n_0 \rangle},$$

where $n_0 \in N$ is primitive with $\langle n_0, m_0 \rangle = 0$ chosen with the following sign convention:

$$\langle n_0, \cdot \rangle < 0$$

$$\langle n_0, \cdot \rangle > 0$$
To such a path, we can associate a path-ordered product of automorphisms, as follows.

First, when $\gamma$ crosses an element $(\partial, f_\partial)$, we obtain an element of $H(R)$ given by

$$\mathbb{R}^m \mapsto \mathbb{R}^m f_\partial \langle m, n_0 \rangle,$$

where $n_0 \in N$ is primitive with $\langle n_0, m_0 \rangle = 0$ chosen with the following sign convention:

This defines an element $\theta_{\gamma, \partial} \in H(R)$. 
The path-ordered product is then defined by

$$\theta_{\mathcal{D},\gamma} = \prod \theta_{\mathcal{D},\gamma},$$

where the product runs over all $\mathcal{D}$ crossed by $\gamma$, in the order traversed by $\gamma$. 
Example: Commutators I

$$\mathfrak{D} = \{(\varnothing_1, f_1), (\varnothing_2, f_2)\},$$

where $\varnothing_1$, $\varnothing_2$ are lines through the origin.
**Example: Commutators I**

\[ \mathcal{D} = \{ (\mathfrak{v}_1, f_1), (\mathfrak{v}_2, f_2) \}, \]

where \( \mathfrak{v}_1, \mathfrak{v}_2 \) are lines through the origin.

Then

\[ \theta_{\mathcal{D}, \gamma} = \theta_2^{-1} \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1, \]

where \( \theta_1 \) and \( \theta_2 \) are the elements of \( H(R) \) associated to the first two crossings.
**Kontsevich-Soibelman Lemma.** Given a scattering diagram $\mathcal{D}$, there is a scattering diagram $\mathcal{D}'$ containing $\mathcal{D}$ such that $\mathcal{D}' \setminus \mathcal{D}$ consists only of rays, and

$$\theta_{\mathcal{D}',\gamma} = id$$

for every closed loop $\gamma$ for which $\theta_{\mathcal{D}',\gamma}$ is defined.
Example: Commutators II

$$\mathcal{D}$$

\[ x \mapsto x(1 + ty^{-1}) \]

\[ y \mapsto y \]

\[ x \mapsto x \]

\[ y \mapsto y(1 + tx^{-1}) \]

\[ y \mapsto y/(1 + tx^{-1}) \]

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Example: Commutators II

\[ D' \]
\begin{align*}
x & \mapsto x(1 + ty^{-1}) \\
y & \mapsto y
\end{align*}

\[ y \mapsto y/\left(1 + t^2 x^{-1} y^{-1}\right) \]

\[ x \mapsto x \]
\[ y \mapsto y/\left(1 + tx^{-1}\right) \]

\[ x \mapsto x/(1 + ty^{-1}) \]
\[ y \mapsto y \]
Example: Commutators III

\[ (1 + tx^{-1})^2 \]

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Example: Commutators III

Lines of slope \((n + 1)/n, n \geq 1\): \((1 + t^{2n+1}x^{-n}y^{-n-1})^2\)

Lines of slope \(n/(n + 1), n \geq 1\): \((1 + t^{2n+1}x^{-n-1}y^{-n})^2\)

Line of slope 1:

\[
\left(1 - t^2x^{-1}y^{-1}\right)^{-4} = \frac{(1 + t^2x^{-1}y^{-1})^4}{(1 - t^4x^{-2}y^{-2})^{2\cdot2}}.
\]
Example: Commutators IV

$$\mathcal{D}$$

$$\gamma$$

$$(1 + tx^{-1})^3$$

$$(1 + ty^{-1})^3$$
Example: Commutators IV

Have rays of slope $3, \frac{8}{3}, \frac{21}{8}, \ldots$ converging to $(3 + \sqrt{5})/2$.
Have rays of slope $1/3, \frac{3}{8}, \frac{8}{21}, \ldots$ converging to $(3 - \sqrt{5})/2$.
Have rays of all rational slopes between $(3 - \sqrt{5})/2$ and $(3 + \sqrt{5})/2$. 
Functions attached to rays are complicated.
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For example, the function attached to the line of slope 1 is

\[
\left( \sum_{n=0}^{\infty} \frac{1}{3n+1} \binom{4n}{n} t^{2n} x^{-n} y^{-n} \right)^9
= \frac{(1 + t^2 x^{-1} y^{-1})^9 \cdot (1 + t^6 x^{-3} y^{-3})^{3.54} \ldots}{(1 - t^4 x^{-2} y^{-2})^{2.18} \cdot (1 - t^8 x^{-4} y^{-4})^{4.252} \ldots}
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\]

**Remark.** This description is based on computer calculations and may not yet have been verified.
3. The tropical vertex, A-model

Consider a weighted projective space $X$ given by the fan:

The three labelled rays correspond to three toric divisors, $D_1, D_2,$ and $D_{out}$.
Pick two integers $d_1, d_2 > 0$ and general sets of points

\[ S_1 \subseteq D_1, \]
\[ S_2 \subseteq D_2 \]

with

\[ \#S_1 = d_1, \#S_2 = d_2, \]
Pick two integers \( d_1, d_2 > 0 \) and general sets of points
\[
S_1 \subseteq D_1, \\
S_2 \subseteq D_2
\]
with
\[
\#S_1 = d_1, \#S_2 = d_2,
\]

**Definition.** Let \( N_d \) be the number of maps \( \varphi : \mathbb{P}^1 \to X \) (up to reparametrization) satisfying the following properties:

1. Whenever \( \varphi(p) \in D_i, \ i = 1, 2, \) then \( \varphi(p) \in S_i \) and \( \varphi \) is transversal to \( D_i \) at \( \varphi(p) \).
2. There is a unique \( q \in \mathbb{P}^1 \) such that \( \varphi(q) \in D_{out} \).
3. The intersection multiplicity of \( \varphi(\mathbb{P}^1) \) with \( D_{out} \) at \( \varphi(q) \) is \( d \).
Remark. This can be defined more precisely as a sum of Gromov-Witten invariants over certain homology classes on the blow-up $\tilde{X}$ of $X$ along the set $S_1 \cup S_2$ with the reduced scheme structure. Furthermore, we need to keep in mind there may be multiple cover contributions.
**Examples.** \( d_1 = d_2 = 1, (a, b) = (1, 1). \)
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**Examples.** $d_1 = d_2 = 1, (a, b) = (1, 1)$.

\[ N_1 = 1, \ N_d = 0, \ d \geq 2. \]
**Examples.** $d_1 = d_2 = 2$, $(a, b) = (1, 1)$. 
Examples. \( d_1 = d_2 = 2, (a, b) = (1, 1). \)

\( N_1 = 4 \)
Examples. $d_1 = d_2 = 2$, $(a, b) = (1, 1)$.

$N_1 = 4 \quad N_2 = 2$, $N_d = 0$, $d \geq 3$
Examples. $d_1 = d_2 = 2$, $(a, b) = (1, 1)$.

$N_1 = 4$ \quad $N_2 = 2$, $N_d = 0$, $d \geq 3$

Compare with the function attached to the ray of slope 1:

$$\frac{(1 + t^2 x^{-1} y^{-1})^4}{(1 - t^4 x^{-2} y^{-2})^{2.2}}$$
Examples. $d_1 = d_2 = 3$, $(a, b) = (1, 1)$. 

\[ D_{\text{out}} \]
Examples. \( d_1 = d_2 = 3, \ (a, b) = (1, 1). \)
Examples. $d_1 = d_2 = 3$, $(a, b) = (1, 1)$.

$N_1 = 9$, $N_2 = 3 \times 3 \times 2 = 18$
Examples. $d_1 = d_2 = 3, (a, b) = (1, 1)$. 

$N_1 = 9, \ N_2 = 3 \times 3 \times 2 = 18$

18 such cubics.
Examples. $d_1 = d_2 = 3$, $(a, b) = (1, 1)$.

$N_1 = 9$, $N_2 = 3 \times 3 \times 2 = 18$, $N_3 = 18 + 36 = 54, \ldots$

$2 \times 3 \times 2 \times 3 = 36$ such cubics
Summary. \( d_1 = d_2 = 3, (a, b) = (1, 1). \)

\[ N_1 = 9, N_2 = 18, N_3 = 54, N_4 = 252, \ldots, \]
Section 3: The tropical vertex, A-model

Summary. \( d_1 = d_2 = 3, (a, b) = (1, 1). \)

\( N_1 = 9, N_2 = 18, N_3 = 54, N_4 = 252, \ldots, \)

Compare with

\[
\frac{(1 + t^2 x^{-1} y^{-1})^9 \cdot (1 + t^6 x^{-3} y^{-3})^{3 \cdot 54} \ldots}{(1 - t^4 x^{-2} y^{-2})^{2 \cdot 18} \cdot (1 - t^8 x^{-4} y^{-4})^{4 \cdot 252} \ldots}
\]
Conjecture. Let $\mathcal{D}$ be the scattering diagram consisting of two lines with attached functions $(1 + tx^{-1})^{d_1}$ and $(1 + ty^{-1})^{d_2}$ and let $\mathcal{D}'$ be the scattering diagram obtained from the Kontsevich-Soibelman Lemma. Then the function $f_{out}$ attached to the ray generated by a primitive vector $(a, b)$ satisfies

$$\log f_{out} = \sum_{d=1}^{\infty} dN_d t^{d(a+b)} x^{-da} y^{-db}.$$
In addition, there are multiple cover formulae: the contribution to multiple covers of a “nice” curve which is $d$-tangent to $D_{out}$ is

$$\sum_{d=1}^{\infty} kd \left( \frac{(d - 1)k - 1}{k - 1} \right) t^{dk(a+b)} x^{-kda} y^{-kdb} \frac{1}{k^2}.$$ 

If $d = 1$, we interpret $\left( \frac{-1}{k - 1} \right) = (-1)^{k-1}$, giving

$$\sum_{k=1}^{\infty} k (-1)^{k-1} \frac{t^k(a+b) x^{-ka} y^{-kb}}{k^2}.$$
We can rewrite the formula for $f_{\text{out}}$ as

$$f_{\text{out}} = \prod_{d=1}^{\infty} G_d(t^{a+b}x^{-a}y^{-b})I_d$$

where

$$G_d(q) = \left( \sum_{k=0}^{\infty} \frac{1}{(d-2)k+1} \binom{d-1}{k} q^{kd} \right)^d$$

Here

$$G_1(q) = 1 + q$$
$$G_2(q) = \frac{1}{(1 - q^2)^2}$$