Math 53 – Practice Final – Solutions

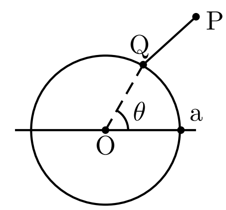
1. $P:(1,1,-1)$, $Q:(1,2,0)$, $R:(-2,2,2)$, so $\overrightarrow{PQ}=\langle 0,1,1\rangle$ and $\overrightarrow{PR}=\langle -3,1,3\rangle$. Thus

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ -3 & 1 & 3 \end{vmatrix}=\langle 2,-3,3\rangle.$$

The vector $\langle 2,-3,3\rangle$ is normal to the plane through P,Q,R. Plugging any of the given points into the equation $2x-3y+3z=d$, we obtain:

$$2x-3y+3z=-4.$$

2. $\overrightarrow{OP}=\overrightarrow{OQ}+\overrightarrow{QP}$, where $\overrightarrow{OQ}=\langle a\cos\theta, a\sin\theta\rangle$, and $\overrightarrow{QP}=a\theta\,\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle$.
Hence $x=a\cos\theta+\dfrac{a\theta}{\sqrt{2}}$, $y=a\sin\theta+\dfrac{a\theta}{\sqrt{2}}$.



3. a) $\vec{r}=\langle 3\cos t, 5\sin t, 4\cos t\rangle \Rightarrow \vec{v}=d\vec{r}/dt=\langle -3\sin t, 5\cos t, -4\sin t\rangle$, and $|\vec{v}|=\sqrt{9\sin^2 t+16\sin^2 t+25\cos^2 t}=5$.

b) The trajectory passes through the yz-plane when $x=0$, hence when $\cos t=0$, i.e. $t=\pi/2$ and $3\pi/2$. Thus, the intersections occur at the points $(0,\pm 5,0)$.

4. $w=x^2y-xy^3$, and $P=(2,1)$:

a) $\nabla w=\langle 2xy-y^3, x^2-3xy^2\rangle$, so $\nabla w(P)=\langle 3,-2\rangle$. The unit vector in the direction of $\vec{A}=\langle 3,4\rangle$ is $\hat{u}=\dfrac{\vec{A}}{|\vec{A}|}=\dfrac{\langle 3,4\rangle}{5}$. So $D_{\hat{u}}w=\nabla w\cdot\hat{u}=\langle 3,-2\rangle\cdot\dfrac{\langle 3,4\rangle}{5}=\dfrac{1}{5}$.

b) $D_{\hat{u}}w=\dfrac{1}{5}\approx\dfrac{\Delta w}{\Delta s}$, so $\Delta w\approx\dfrac{1}{5}\Delta s=\dfrac{1}{5}(0.01)=0.002$.

5. a) Let $g(x,y,z)=x^2+2y^2+2z^2$: then $\nabla g=\langle 2x,4y,4z\rangle=\langle 2,4,4\rangle$ at $(1,1,1)$. Since ∇g is normal to the tangent plane, we get the equation: $2x+4y+4z=10$, or $x+2y+2z=5$.

b) Dihedral angle (angle between normals): $\cos\theta=\dfrac{\langle 1,2,2\rangle\cdot\langle 0,0,1\rangle}{(\sqrt{1^2+2^2+2^2})(1)}=\dfrac{2}{3}$. So $\theta=\cos^{-1}(2/3)$.

6. $f(x,y)=x^2+xy+y^2-4x-5y+7$: the critical points correspond to $f_x=2x+y-4=0$ and $f_y=x+2y-5=0$. Solving, we get $x=1$ and $y=2$. So the only critical point is $(1,2)$. Moreover $f(1,2)=0$.

Next we check the boundaries and infinity. On the x-axis: $f(x,0)=x^2-4x+7=(x-2)^2+3>0$; on the y-axis: $f(0,y)=y^2-5y+7=(y-\frac{5}{2})^2+\frac{3}{4}>0$. At infinity: if x and/or y tends to $+\infty$ then $f(x,y)\to+\infty$. So the minimum of f in the first quadrant is at $(1,2)$.

(Note: the second derivative test shows that $(1,2)$ is a local minimum, but this is not sufficient to conclude regarding the absolute minimum.)

7. Minimize $f(x,y,z)=x^2+y^2+z^2$ with constraint $g(x,y,z)=2x+y-z=6$: the Lagrange equations ($\nabla f=\lambda\nabla g$) are: $2x=2\lambda$, $2y=\lambda$, $2z=-\lambda$. Substituting into the constraint equation: $2x+y-z=2\lambda+\frac{\lambda}{2}-(-\frac{\lambda}{2})=3\lambda=6$. So $\lambda=2$, and $(x,y,z)=(2,1,-1)$.

8. a) $f(x,y,z)=x$; the constraint is $g(x,y,z)=x^4+y^4+z^4+xy+yz+zx=6$. The Lagrange multiplier equation is:

$$\nabla f=\lambda\nabla g \quad \Leftrightarrow \quad \left\{\begin{array}{rcl} 1 &=& \lambda(4x^3+y+z) \\ 0 &=& \lambda(4y^3+x+z) \\ 0 &=& \lambda(4z^3+x+y) \end{array}\right.$$

b) The level surfaces of f and g are tangent at (x_0,y_0,z_0), so they have the same tangent plane. The level surface of f is the plane $x=x_0$; hence this is also the tangent plane to the surface $g=6$ at (x_0,y_0,z_0).

Second method: at (x_0,y_0,z_0), we have $\left.\begin{array}{r} 1= \lambda g_x \\ 0= \lambda g_y \\ 0= \lambda g_z \end{array}\right\} \quad\Rightarrow\quad \lambda\neq 0$ and $\langle g_x,g_y,g_z\rangle=\langle\frac{1}{\lambda},0,0\rangle$.

So $\langle\frac{1}{\lambda},0,0\rangle$ is perpendicular to the tangent plane at (x_0,y_0,z_0); the equation of the tangent plane is then $\frac{1}{\lambda}(x-x_0)=0$, or equivalently $x=x_0$.

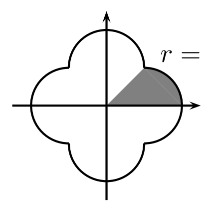
9. At the point P, differentiating the constraint gives: $dg=g_x\,dx+g_y\,dy+g_z\,dz=0$, so $dz=-\frac{g_x}{g_z}\,dx-\frac{g_y}{g_z}\,dy$. Hence $\frac{\partial z}{\partial x}=-\frac{g_x}{g_z}=-\frac{2}{-1}=2$ (using the given values of g_x and g_z.)

10. The region of integration is bounded by the parabola $y=x^2$ (or $x=\sqrt{y}$), the horizontal line $y=9$, and the y-axis. So:

$$\int_0^3\int_{x^2}^9 xe^{-y^2}\,dy\,dx=\int_0^9\int_0^{\sqrt{y}} xe^{-y^2}\,dx\,dy.$$

Inner: $\left[\frac{1}{2}x^2e^{-y^2}\right]_0^{\sqrt{y}}=\frac{1}{2}ye^{-y^2}$. Outer: $\left[-\frac{1}{4}e^{-y^2}\right]_0^9=\frac{1}{4}(1-e^{-81})$.

11. By symmetry, we integrate over 1/8 of the region; recalling that the polar moment of inertia is $I_0=\iint_R r^2\,\rho\,dA$ (here $\rho=1$), we get



$$8\int_0^{\pi/4}\int_0^{2\cos\theta} r^2\,r\,dr\,d\theta \qquad \left(\text{or } 4\int_{-\pi/4}^{\pi/4}\cdots\right)$$

12. a) $\hat{\mathbf{n}}\,ds=\langle dy,-dx\rangle$, so Flux $=\int_C\vec{F}\cdot\hat{\mathbf{n}}\,ds=\int_C -Q\,dx+P\,dy$.

b) By Green's theorem, $\oint_C -Q\,dx+P\,dy=\iint_R(P_x+Q_y)\,dA=\iint_R(a+b)\,dA=(a+b)\,\text{area}(R)$. This equals the area of R if and only if $a+b=1$.

13. $\bar{z}=\frac{1}{\text{Volume}}\iiint z\,dV$, since here the density is 1. The volume is $\frac{2}{3}\pi$ (half of the unit sphere), and in spherical coordinates $z=\rho\cos\phi$. So $\bar{z}=\frac{3}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Inner: $\int_0^1\rho^3\cos\phi\sin\phi\,d\rho=\left[\frac{1}{4}\rho^4\cos\phi\sin\phi\right]_0^1=\frac{1}{4}\cos\phi\sin\phi$.
Middle: $\int_0^{\pi/2}\frac{1}{4}\sin\phi\,\cos\phi\,d\phi=\left[\frac{1}{8}\sin^2\phi\right]_0^{\pi/2}=\frac{1}{8}$. Outer: $\frac{3}{2\pi}\cdot 2\pi\cdot\frac{1}{8}=\frac{3}{8}$.

14. The line from $P:(1,1,1)$ to $Q:(2,4,8)$ has parametric equations: $x=1+t$, $y=1+3t$, $z=1+7t$ (since $\overrightarrow{PQ}=\langle 1,3,7\rangle$). The line segment corresponds to $0\le t\le 1$. So

$\int_C(y-x)\,dx+(y-z)\,dz=\int_0^1 2t\,dt+(-4t)\,7\,dt=\int_0^1-26t\,dt=\left[-13t^2\right]_0^1=-13$.

15. a) $\vec{F} = \langle P, Q, R\rangle = \langle ay^2, 2yx + 2yz, by^2 + z^2\rangle$: we need $P_y = 2ay = Q_x = 2y$, so $a = 1$; and $P_z = 0 = R_x$; and $Q_z = 2y = R_y = 2by$, so $b = 1$. So: $\vec{F}$ is conservative when $a = 1$ and $b = 1$.

b) $f_x = y^2$, so $f(x, y, z) = xy^2 + g(y, z)$. Differentiating wrt y, $f_y = 2xy + g_y = 2xy + 2yz$.
So $g_y = 2yz$, hence $g(y, z) = y^2 z + h(z)$ and $f(x, y, z) = xy^2 + y^2 z + h(z)$.
Differentiating wrt z, $f_z = y^2 + h'(z) = y^2 + z^2$ so $h'(z) = z^2$, hence $h(z) = \frac{1}{3}z^3 + c$.
Finally we get: $f(x, y, z) = xy^2 + y^2 z + \frac{1}{3}z^3 + c$.

c) any surface S of the form $xy^2 + y^2 z + \frac{1}{3}z^3 = K$ for some constant K (i.e. a level surface of f). Then by the fundamental theorem, $\int_P^Q \vec{F} \cdot d\vec{r} = f(Q) - f(P) = 0$ if P and Q lie on S.

16. $\displaystyle\iint_B \vec{F}\cdot\hat{\mathbf{n}}\,dS + \iint_U \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_S \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_R \operatorname{div}\vec{F}\,dV = \iiint_R 3\,dV = 3\,vol(V)$, where

$$vol(V) = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^1 (1-r^2)\,r\,dr = 2\pi\left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = \pi/2.$$

So the total flux through B and U equals $3\pi/2$. Next we compute directly the flux through the bottom disc B, where $z = 0$ and $\hat{\mathbf{n}} = -\hat{\mathrm{k}}$:
$\iint_B \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_B \langle x, y, z\rangle\cdot\langle 0, 0, -1\rangle\,dx\,dy = \iint_B -z\,dx\,dy = \iint_B 0\,dS = 0$.

Hence $\iint_U \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_S \vec{F}\cdot\hat{\mathbf{n}}\,dS = 3\pi/2$.

17. U is the graph $z = f(x, y) = 1 - x^2 - y^2$, so $\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle dx\,dy = \langle 2x, 2y, 1\rangle dx\,dy$.

So $\displaystyle\iint_U \vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_U \langle x, y, z\rangle\cdot\langle 2x, 2y, 1\rangle dx\,dy = \iint_U (2x^2 + 2y^2 + z)\,dx\,dy.$

Recalling that $z = 1 - x^2 - y^2$ on U, this is equal to

$$\iint_U (x^2 + y^2 + 1)\,dx\,dy = \int_0^{2\pi}\int_0^1 (r^2 + 1)\,r\,dr\,d\theta = 2\pi\left[\tfrac{1}{4}r^4 + \tfrac{1}{2}r^2\right]_0^1 = 2\pi\cdot\frac{3}{4} = 3\pi/2.$$

18. By Stokes, if S_1 is the portion of S enclosed by C, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1}(\nabla\times\vec{F})\cdot\hat{\mathbf{n}}\,dS$. Here

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathrm{k}} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & xz \end{vmatrix} = \langle 0, -z, 0\rangle, \quad \text{and } \hat{\mathbf{n}}\,dS = \left\langle -\tfrac{\partial z}{\partial x}, -\tfrac{\partial z}{\partial y}, 1\right\rangle\,dx\,dy = \langle -f'(x), 0, 1\rangle dx\,dy$$

(since S_1 is a graph $z = f(x)$). So $\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1}(\nabla\times\vec{F})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} 0\,dx\,dy = 0$.

19. a) C is the circle $x^2 + y^2 = 1$, $z = 1$. Parametrization: $x = \cos t$, $y = \sin t$, $z = 1$; and $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = 0$. So

$I = \oint_C xz\,dx + y\,dy + y\,dz = \int_0^{2\pi} \cos t(-\sin t)\,dt + \sin t\cos t\,dt + 0\,dt = 0$.

b) $$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathrm{k}} \\ \partial_x & \partial_y & \partial_z \\ xz & y & y \end{vmatrix} = \hat{\imath} + x\hat{\jmath}.$$

c) By Stokes' theorem, $I = \oint_C \vec{F}\cdot d\vec{r} = \iint_S(\nabla\times\vec{F})\cdot\hat{\mathbf{n}}\,dS = \iint_S(\hat{\imath} + x\hat{\jmath})\cdot\hat{\mathbf{n}}\,dS$.

Note: $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\langle x, y, z\rangle$, so $\vec{F}\cdot\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(x + xy)$ (where $x = \sqrt{2}\sin\phi\cos\theta$ and $y = \sqrt{2}\sin\phi\sin\theta$); and $dS = 2\sin\phi\,d\phi\,d\theta$.