Math 53 Homework 11

Due Wednesday 4/13/16 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 4/4: Gradient fields, fundamental theorem for line integrals

- **Read:** section 16.3.
- **Work:** 16.3: (3), (5), 7, (10), (11), (13), 15, (17), 19[1], (23), 25, (29).[2]
  Problems 1 and 2 below.

Wednesday 4/6: Green's theorem

- **Read:** section 16.4.
- **Work:** 16.4: (1), (2), 3, (4), (7), 9, (11), 13[3], (17), 19, (21), 25, (26).
  Problem 3 below.

Friday 4/8: Curl and divergence; vector forms of Green's theorem

- **Read:** section 16.5.
- **Work:** 16.5: (1), (7), 9, 11, (12), 13, (16), (21), 25, (26), (32), 33, (34), 36*, 37.
  Problems 4 and 5 below. (Problem 5 due with HW 12)

* Remark: 16.5 # 36 shows that, if a harmonic function f on a domain D is zero everywhere at its boundary curve C, then $f = 0$ everywhere in D. A consequence is that, if f_1 and f_2 are two harmonic functions in D such that $f_1 = f_2$ at every point of C, then $f_1 = f_2$ everywhere in D (by applying the previous result to the harmonic function $f_1 - f_2$). This uniqueness property has important applications in mathematical analysis.

Problem 1. The goal of this problem is to show the importance of the condition that the domain under consideration be a simply connected region (i.e., without holes) in the criterion for a vector field to be conservative.

Consider the vector field $\vec{F}(x,y) = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2}$.

a) Show that $\vec{F}$ is the gradient of the polar angle function $\theta(x,y) = \tan^{-1}(y/x)$ defined over the right half-plane $x > 0$. (Note: this formula for θ does not make sense for $x = 0$!)

[1] **6th/7th ed:** do the 8th ed problem: $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$, C any path from (1,0) to (2,1).
[2] **6th ed:** 16.3: (3), (5), 7, (10), (11), (13), 15, (17), 19[1], (21), 23, (27). **7th ed:** same as 8th.
[3] **6th ed:** do the 7th/8th ed problem: $\vec{F}(x,y) = \langle y - \cos y, x\sin y\rangle$, C is the circle $(x-3)^2+(y+4)^2 = 4$ oriented clockwise.

b) Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Express $\int_C \vec{F} \cdot d\vec{r}$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B.

c) Compute directly from the definition the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_1 is the upper half of the unit circle running from $(1, 0)$ to $(-1, 0)$, and C_2 is the lower half of the unit circle, also going from $(1, 0)$ to $(-1, 0)$.

d) Using the results of parts (a)-(c), is $\vec{F}$ conservative (path-independent) over its entire domain of definition? Is it conservative over the right half-plane $x > 0$? Justify your answers.

e) Show that the components P and Q of $\vec{F}$ satisfy the equation $\partial P/\partial y = \partial Q/\partial x$ at any point of the plane where $\vec{F}$ is defined (not just in the right half-plane $x > 0$).

f) Using Green's theorem, show that $\int_C \vec{F} \cdot d\vec{r} = 0$ for every simple closed curve that does not pass through or enclose the origin. Does this remain true if C encloses the origin?

Note: in fact it is true that $\vec{F} = \nabla\theta$ everywhere. However, the polar angle θ cannot be defined as a single-valued differentiable function everywhere (if you try, you will find that it is only well-defined up to adding multiples of 2π). This is why in parts (a) and (b) we only consider the right half-plane; any other region over which θ can be defined unambiguously in a continuous manner would be equally suitable.

Problem 2.

a) For which values of n do the components P and Q of $\vec{F} = r^n(x\,\hat{\imath} + y\,\hat{\jmath})$ satisfy $\partial P/\partial y = \partial Q/\partial x$? (Here $r = \sqrt{x^2 + y^2}$; start by finding formulas for r_x and r_y).

b) Whenever possible, find a function g such that $\vec{F} = \nabla g$. (Hint: look for a function of the form $g = g(r)$, with $r = \sqrt{x^2 + y^2}$. Watch out for a certain negative value of n for which the general formula doesn't work.)

Problem 3. This problem shows how the value of the integral $I_n = \displaystyle\int_0^{2\pi} \cos^{2n}\theta\, d\theta$ can be determined for all n using Green's theorem (the conventional method is via integration by parts).

a) Prove that $I_n = \dfrac{2n-1}{2n} I_{n-1}$ by writing Green's theorem for the line integral $\oint_C x^{2n-1}\, dy$, where C is the unit circle $x^2 + y^2 = 1$ counterclockwise, and by evaluating each of the integrals separately until it looks like either I_n or I_{n-1}.

b) What is the value of I_0? Using the result of (a), find expressions for I_1, I_2, I_3 (don't simplify fractions or calculate products), then give a general formula for I_n.

Problem 4.

a) Let C be the unit circle, oriented counterclockwise, and consider the vector field $\vec{F} = x^2\,\hat{\imath} + xy\,\hat{\jmath}$. Describe geometrically and/or sketch the vector field $\vec{F}$ (see also 16.2 # 32 (b) assigned on HW 10). Which portions of C contribute positively to the flux $\int_C \vec{F} \cdot \hat{n}\, ds$? Which portions contribute negatively?

b) Find the flux of $\vec{F}$ through C by directly evaluating the line integral $\int_C \vec{F} \cdot \hat{n}\, ds$.

Explain your answer using (a).

c) Find the flux of $\vec{F}$ through C by using the second vector form ("normal form") of Green's theorem.

Problem 5. – DUE WITH HW 12

Find the flux of the vector field $\vec{F} = \dfrac{x\hat{\imath} + y\hat{\jmath}}{x^2 + y^2}$ outwards through the circle centered at $(1, 0)$ of radius $a \neq 1$. Consider the cases $a > 1$ and $a < 1$ separately, and use Green's theorem (carefully!). Explain your answers with diagrams.