Math 53 – Practice Midterm 2 A – 80 minutes

Problem 1. (8 points) Let \((\bar{x}, \bar{y})\) be the center of mass of the triangle with vertices at \((-2, 0), (0, 1), (2, 0)\) and uniform density \(\rho = 1\).

Write an integral formula for \(\bar{y}\). Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.

Problem 2. (8 points) Find the polar moment of inertia \(I_0 = \iint (\rho^2) \, dA\) of the unit disk with density \(\rho\) equal to the distance from the \(y\)-axis.

Problem 3. (7 points) For \(\vec{F} = yx^3 \hat{i} + y^2 \hat{j}\), find \(\int_C \vec{F} \cdot d\vec{r}\) on the portion of the parabola \(y = x^2\) from \((0, 0)\) to \((1, 1)\).

Problem 4. (10 points) Consider the vector field \(\vec{F} = (ax^2 y + y^3 + 1) \hat{i} + (2x^3 + bxy^2 + 2) \hat{j}\), where \(a\) and \(b\) are constants.

a) (3) Find the values of \(a\) and \(b\) for which \(\vec{F}\) is conservative.

b) (4) For these values of \(a\) and \(b\), find \(f(x, y)\) such that \(\vec{F} = \nabla f\). (Use a systematic method and show your work.)

c) (3) Still using the values of \(a\) and \(b\) from part (a), compute \(\int_C \vec{F} \cdot d\vec{r}\) along the curve \(C\) given by the parametric equations \(x = e^t \cos t, y = e^t \sin t, 0 \leq t \leq \pi\).

Problem 5. (10 points) Consider the region \(R\) in the first quadrant bounded by the curves \(y = x^2, y = x^2/5, xy = 2, \text{ and } xy = 4\).

a) (5) Compute \(dxdy\) in terms of \(dudv\) if \(u = x^2/y\) and \(v = xy\).

b) (5) Express the area of \(R\) as a double integral in \(uv\) coordinates and evaluate it.

Problem 6. (7 points)

a) (3) Let \(C\) be a simple closed curve going counterclockwise around a region \(R\). Let \(M = M(x, y)\). Express \(\int_C M \, dx\) as a double integral over \(R\).

b) (4) Find \(M\) so that \(\int_C M \, dx\) is the mass of \(R\) with density \(\rho(x, y) = (x + y)^2\).

Problem 7. (15 points) Consider the region \(R\) enclosed by the \(x\)-axis, \(x = 1\) and \(y = x^3\).

a) (5) Use Green’s theorem to find the flux \(\oint \vec{F} \cdot \hat{n} \, ds\) of \(\vec{F} = (1 + y^2) \hat{j}\) out of \(R\).

b) (7) Find the flux of \(\vec{F}\) out of \(R\) through the two sides \(C_1\) (the horizontal segment) and \(C_2\) (the vertical segment).

c) (3) Use parts (a) and (b) to find the flux out of the third side \(C_3\).
Problem 8. (8 points) Let \( C \) be the portion of the cylinder \( x^2 + y^2 \leq 1 \) lying in the first octant \((x \geq 0, \, y \geq 0, \, z \geq 0)\) and below the plane \( z = 1 \). Set up a triple integral in cylindrical coordinates which gives the moment of inertia of \( C \) about the \( z \)-axis; assume the density to be \( \delta = 1 \). (Recall \( I_z = \iiint (x^2 + y^2) \, \delta \, dV \).)

(Give integrand and limits of integration, but do not evaluate.)

Problem 9. (10 points)
A solid sphere \( S \) of radius \( a \) is placed above the \( xy \)-plane so it is tangent at the origin and its diameter lies along the \( z \)-axis. Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere \( S \) lying above the plane \( z = a \). (Give integrand and limits of integration, but do not evaluate.)

Problem 10. (17 points) Let \( S \) be the surface formed by the portion of the paraboloid \( z = 1 - x^2 - y^2 \) lying above the \( xy \)-plane, and let \( \vec{F} = x \vec{i} + y \vec{j} + 2(1-z) \vec{k} \).

Calculate the flux of \( \vec{F} \) across \( S \), taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) (10) by direct calculation of \( \iint_S \vec{F} \cdot \hat{n} \, dS \);

b) (7) by computing the flux of \( \vec{F} \) across a simpler surface and using the divergence theorem.