14.5 # 5: \[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 2t e^{y/z} - \frac{x}{z^2} e^{y/z} - 2 \frac{x y}{z^2} e^{y/z} = \left(2t - \frac{t^2}{1 + 2t} - \frac{2t^2 (1 - t)}{(1 + 2t)^2}\right) e^{(1-t)/(1+2t)}.
\]

14.5 # 7: \[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 5(x-y)^4(2st) - 5(x-y)^4 t^2 = 5(x-y)^4(2st - t^2).
\]
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 5(x-y)^4 s^2 - 5(x-y)^4(2st) = 5(x-y)^4(s^2 - 2st).
\]

14.5 # 15: \(g(u,v) = f(x(u,v), y(u,v))\) where \(x(u,v) = e^u + \sin v\) and \(y(u,v) = e^u + \cos v\). So \(g_u = f_x x_u + f_y y_u = e^u f_x + e^u f_y\), and \(g_v = f_x x_v + f_y y_v = \cos v f_x - \sin v f_y\). At \((u,v) = (0,0), (x,y) = (1,2);\) so \(g_u(0,0) = 1 \cdot f_x(1,2) + 1 \cdot f_y(1,2) = 2 + 5 = 7\), and \(g_v(0,0) = 1 \cdot f_x(1,2) - 0 \cdot f_y(1,2) = 2\).

14.5 # 18: \[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u};
\]
\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}.
\]

14.5 # 43: Call the sides \(a\) and \(b\), and the angle between them \(\theta\); so the area \(A = \frac{1}{2} ab \sin \theta\) is assumed to be constant. So \(\frac{dA}{dt} = \frac{\partial A}{\partial a} \frac{da}{dt} + \frac{\partial A}{\partial b} \frac{db}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} = 0\), i.e., \(\frac{1}{2} b \sin \theta \frac{da}{dt} + \frac{1}{2} a \sin \theta \frac{db}{dt} + \frac{1}{2} ab \cos \theta \frac{d\theta}{dt} = 0\). Solving for \(d\theta/dt\), we get:
\[
\frac{d\theta}{dt} = -\frac{b \sin \theta \frac{da}{dt} + a \sin \theta \frac{db}{dt}}{ab \cos \theta}.
\]
Using the given values for \(a, b, \theta, da/dt, db/dt,\) we get \(\frac{d\theta}{dt} = -\frac{30 \cdot \frac{1}{2} \cdot 3 + 20 \cdot \frac{1}{2} \cdot (-2)}{20 \cdot 30 \cdot \frac{\sqrt{3}}{2}} = -\frac{5}{60 \sqrt{3}} \approx -0.048 \text{ rad/s.}\)

14.5 # 45: (a) chain rule: \(\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta, \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} r \cos \theta.\)

(b) \(\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta,\) and
\[
\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2 \theta - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} r^2 \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2 \theta.\) So
\[
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \left(\cos^2 \theta + \sin^2 \theta\right) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.
\]

14.5 # 51: \(\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2s \frac{\partial z}{\partial x} + 2r \frac{\partial z}{\partial y}.\)
\[
\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial}{\partial r} \left(2s \frac{\partial z}{\partial x} + 2r \frac{\partial z}{\partial y}\right) = 2s \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x}\right) + 2r \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y}\right) + 2 \frac{\partial z}{\partial y}.
\]

\[
= 2s \left(\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x}\right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y}\right) \frac{\partial y}{\partial r}\right) + 2r \left(\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x}\right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y}\right) \frac{\partial y}{\partial r}\right) + 2 \frac{\partial z}{\partial y}.
\]
\[
\begin{align*}
= 4rs \frac{\partial^2 z}{\partial x^2} + 4s^2 \frac{\partial^2 z}{\partial y \partial x} + 4r^2 \frac{\partial^2 z}{\partial x \partial y} + 4rs \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial y} \\
= 4rs \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) + 4(r^2 + s^2) \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial z}{\partial y}.
\end{align*}
\]

Explanation: to compute \( \frac{\partial^2 z}{\partial r \partial s} \), we differentiate the expression for \( \frac{\partial z}{\partial s} \) with respect to \( r \) (keeping \( s \) constant). The first step involves the product rule. The second one is more subtle. To calculate \( \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial s} \right) \), we use the chain rule more once. If this feels confusing, just set \( g = \frac{\partial z}{\partial x} \) (think of this as some new function of \( x \) and \( y \)), and note that we are trying to calculate \( \frac{\partial g}{\partial y} \). The chain rule gives \( \frac{\partial g}{\partial y} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial y} \). Remembering that \( g = \frac{\partial z}{\partial x} \), the first partials of \( g \) with respect to \( x \) and \( y \) are actually second partial derivatives of \( z \). The chain rule is used similarly to differentiate \( \frac{\partial z}{\partial y} \) with respect to \( r \).

14.6 # 1: We can approximate the directional derivative at \( K \) by the average rate of change of pressure between the points where the line through \( K \) and \( S \) (red on the figure) intersects the contour lines closest to \( K \). In this case we measure that, going about 1/6 of the way towards \( S \), \( \Delta P \) going about 1/6 of the way towards \( S \), \( \Delta s \approx 50 \) km, the pressure drops from 1000 to 996 mb \( (\Delta P \approx -4 \) mb). So \( \Delta q \approx \Delta P \approx \frac{\Delta P}{\Delta s} \approx \frac{-4}{50} = -0.08 \) (millibars per km).

14.6 # 9: (a) \( \nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xyz - y^2z, x^2z - x^3z, x^2y - 3xyz^2 \rangle \).

(b) at \( (x, y, z) = (2, -1, 1) \), \( \nabla f = \langle -3, 2, 2 \rangle \).

(c) \( \hat{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle \) is a unit vector, so \( D_u f = \nabla f \cdot \hat{u} = \langle -3, 2, 2 \rangle \cdot \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle = \frac{2}{5} \).

14.6 # 27: (a) Given a unit direction vector \( \hat{u} \), recall that \( D_u f = \nabla f \cdot \hat{u} = |\nabla f| \cos \theta \) (where \( \theta \) is the angle between \( \nabla f \) and \( \hat{u} \)). Since the minimum value of \( \cos \theta \) is \( -1 \), occurring for \( \theta = \pi \), the minimum value of \( D_u f \) is \( -|\nabla f| \) and occurs when \( \hat{u} \) is in the opposite direction of \( \nabla f \).

(b) \( \nabla f = \langle 4x^2y - 2xy^2, x^4 - 3x^2y^2 \rangle \), so at the point \( (2, -3) \) \( f \) decreases fastest in the direction of \( -\nabla f(2, -3) = -\langle 12, 92 \rangle = \langle -12, 92 \rangle \) (or the corresponding unit vector).

14.6 # 38: \( \nabla f(4, 6) \) is perpendicular to the level curve of \( f \) that passes through \( (4, 6) \); so we sketch a portion of level curve through \( (4, 6) \) (using the nearby level curves as guidelines), and draw a line perpendicular to it. The direction of the gradient vector is parallel to this line, and pointing towards increasing values of \( f \). (towards the lower-right, making about a 65° angle with the horizontal direction).

Next we estimate the magnitude \( |\nabla f| \), which equals the directional derivative of \( f \) at \( (4,6) \) in the direction of \( \nabla f \). We estimate this by finding the average rate of change along the direction perpendicular to the level curve. The points where the line previously drawn intersects the contour lines \( f = -2 \) and \( f = -3 \) are \( \approx 0.5 \) units apart, so \( \Delta f = 1 \) and \( \Delta s \approx 0.5 \), giving \( |\nabla f| \approx \frac{\Delta f}{\Delta s} \approx \frac{1}{0.5} = 2 \). Hence we sketch the gradient vector with length 2. (Diagram omitted).

(Note: we could also have tried to estimate \( f_x \) and \( f_y \) separately and use those to sketch \( \nabla f \); this is much less accurate, especially concerning the direction of \( \nabla f \).)

14.6 # 41: Let \( f(x, y, z) = 2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 \); then we are considering the level surface \( f = 10 \). \( \nabla f = \langle 4(x - 2), 2(y - 1), 2(z - 3) \rangle \), so \( \nabla f(3, 3, 5) = \langle 4, 4, 4 \rangle \).
(a) $\nabla f(3, 5, 5) = \langle 4, 4, 4 \rangle$ is a normal vector for the tangent plane at $(3, 3, 5)$, so an equation of the tangent plane is $4(x - 3) + 4(y - 3) + 4(z - 5) = 0$, or $4x + 4y + 4z = 44$ (or $x + y + z = 11$).

(b) The normal line has direction $\langle 4, 4, 4 \rangle$, so parametric equations are $x = 3 + 4t$, $y = 3 + 4t$, $z = 5 + 4t$. (or using $(1, 1, 1)$, $x = 3 + t$, $y = 3 + t$, $z = 5 + t$).

14.6 # 49: $\nabla f = \langle y, x \rangle$, so $\nabla f(3, 2) = \langle 2, 3 \rangle$. So the tangent line has equation $\langle 2, 3 \rangle \cdot (x - 3, y - 2) = 0 \Rightarrow 2(x - 3) + 3(y - 2) = 0$, which simplifies to $2x + 3y = 12$.

14.6 # 56: first note that the point $(1, 1, 2)$ is on both surfaces. Let $f(x, y, z) = 3x^2 + 2y^2 + z^2$, so the ellipsoid is $f = 9$: then $\nabla f = \langle 6x, 4y, 2z \rangle$, so the tangent plane to the ellipsoid has normal vector $\nabla f(1, 1, 2) = \langle 6, 4, 4 \rangle$, and an equation of the tangent plane is $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is the level surface $g = 0$ where $g(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$, and $\nabla g = \langle 2x - 8, 2y - 6, 2z - 8 \rangle$. So the tangent plane at $(1, 1, 2)$ has normal vector $\nabla g(1, 1, 2) = \langle -6, -4, -4 \rangle$, giving the equation $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. The tangent planes are the same, so the surfaces are tangent to each other at $(1, 1, 2)$.

(Note: it would have been enough to show that the normal vectors are parallel to each other, without determining the equations of the tangent planes.)

**Problem 1.** a) (i) We need the vector $\hat{u} = \langle a, b \rangle$ to be tangent to the level curve through $(\frac{3}{2}, \frac{1}{2})$ (whose shape we can estimate from the neighboring ones). Indeed, the directional derivative $D_u f = \nabla f \cdot \hat{u}$ is zero when $\hat{u}$ is perpendicular to $\nabla f$, i.e. tangent to the level curve through $(\frac{3}{2}, \frac{1}{2})$.

Hence, the two directions in which $df/dt = 0$ are the two unit vectors which are tangent to the level curve at $(\frac{3}{2}, \frac{1}{2})$. (One is about $30^\circ$ counterclockwise from $i$, the other is in the opposite direction i.e. about $150^\circ$ clockwise from $i$).

Estimating the shape of the level curve through $(\frac{3}{2}, \frac{1}{2})$ and drawing its tangent line $L$, we find that $L$ lies on the side of the level curve where $f(x, y) \geq f(\frac{3}{2}, \frac{1}{2})$. So, moving along $L$ in either direction, the value of $f$ reaches a minimum at $(\frac{3}{2}, \frac{1}{2})$.

(ii) The directional derivative is largest in the direction of the gradient $\nabla f$ (perpendicular to level curve, towards larger values of $f$), i.e. about $120^\circ$ counterclockwise from $i$. It is smallest in the direction of $-\nabla f$, i.e. about $60^\circ$ clockwise from $i$. The nearest given level curves ($f = -0.8$ and $f = -1$) are about $0.5$ cm apart, i.e. $0.1$ unit, in that direction, hence we estimate $df/dt \approx \Delta f/\Delta t \approx 0.2/0.1 = 2$ in the direction of $\nabla f$, and $-2$ in the opposite direction.

b) $f_x = 3x^2 - 6x + y + 1$, and $f_y = x - 2y + 1$, so $f_x(\frac{3}{2}, \frac{1}{2}) = -\frac{3}{4}$, $f_y(\frac{3}{2}, \frac{1}{2}) = \frac{3}{2}$. Using the chain rule (or the definition of the directional derivative), $df/dt = f_x dx/dt + f_y dy/dt = -\frac{3}{4} a + \frac{3}{2} b$ when $t = 0$. 

3
(i) We want \(-\frac{3}{4}a + \frac{3}{2}b = 0\), i.e. \(a = 2b\). So we want unit vectors of the form \((2b, b)\).
We need \(|(2b, b)| = \sqrt{5b^2} = 1\), hence \(b = \pm \frac{1}{\sqrt{5}}\). The two solutions are therefore 
\((\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})\) and \((-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})\).

(ii) The largest directional derivative is in the direction of \(\nabla f = (-\frac{3}{4}, \frac{3}{2})\), i.e. the unit vector \((a, b) = \frac{\nabla f}{|\nabla f|} = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})\). The smallest value is in the direction of \(-\nabla f\), i.e. the unit vector \((-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})\). The directional derivatives in those two directions are 
\(\nabla f \cdot (a, b) = \pm |\nabla f| = \pm \frac{3}{\sqrt{5}} \approx \pm 1.68\).

**Problem 2:** \(\nabla f(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)\).

\(\nabla f(3, 6, -2) = \left(\frac{3}{\sqrt{10}}, \frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right)\) gives the direction of maximum rate of change, and the maximum rate is \(D_{\text{dir}(\nabla f)} f = |\nabla f| = 1\).

Since \(f(x, y, z)\) is the distance from the origin to \((x, y, z)\), the answer makes sense geometrically: distance from the origin increases fastest when moving radially outwards, and the rate of increase is 1.

**Problem 3.**

a) \(\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, -6 \rangle = (8, 4, -6)\) at \((4, 2, 3)\). The direction of greatest decrease is that of \(-\nabla g\), i.e. the unit vector \(-\frac{\nabla g}{|\nabla g|} = \langle -8, -4, 6 \rangle = \langle -4, -2, 3 \rangle\).

b) Let \(\Delta x = x - 4\), \(\Delta y = y - 2\), \(\Delta z = z - 3\); then the line in the direction of \((-4, -2, 3)\) can be parametrized by \(\Delta x = -4t\), \(\Delta y = -2t\), \(\Delta z = 3t\). (Dividing by \(\sqrt{29}\) is unnecessary and makes calculations more complicated.) At \(P_0 = (4, 2, 3)\), we have \(g = 2\), and \(\nabla g = \langle 8, 4, -6 \rangle\) (from part (a)), so linear approximation gives

\[g(x, y, z) \approx g(P_0) + \nabla g(P_0) \cdot (\Delta x, \Delta y, \Delta z) = 2 + 8\Delta x + 4\Delta y - 6\Delta z = 2 + 8(-4t) + 4(-2t) - 6(3t) = 2 - 58t.\]

Therefore, \(g = 0\) when \(2 - 58t \approx 0\), or \(t \approx 1/29\). At \(t = 1/29\), \((x, y, z) = (4 - 4t, 2 - 2t, 3 + 3t) = (4 - \frac{4}{29}, 2 - \frac{2}{29}, 3 + \frac{3}{29})\). Evaluating \(g\) at this point, we find \(\approx 0.024\), fairly close to 0.

**14.3 # 75:** \(u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx\), and \(u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin kx\), so indeed \(u_t = \alpha^2 u_{xx}\).

**14.3 # 81:** by product rule and chain rule, \(c(x, t) = (4\pi D)^{-1/2}k^{-1/2}e^{-x^2/(4Dt)} \Rightarrow \)

\[
\frac{\partial c}{\partial t} = -\frac{1}{2} \frac{t^{3/2}}{(4\pi D)^{1/2}} e^{-x^2/(4Dt)} + \frac{x^2}{4Dt^2}(4\pi D)^{-1/2} e^{-x^2/(4Dt)} = \frac{-2Dt + x^2}{8\pi^{1/2} D^{3/2} t^{5/2}} e^{-x^2/(4Dt)},
\]

\[
\frac{\partial c}{\partial x} = \frac{-2x}{4Dt}(4\pi D)^{-1/2} e^{-x^2/(4Dt)} = \frac{-x}{4\pi^{1/2} D^{3/2} t^{5/2}} e^{-x^2/(4Dt)}, \quad \text{and}
\]

\[
\frac{\partial^2 c}{\partial x^2} = \frac{-1 + x(2x/4Dt)}{4\pi^{1/2} D^{3/2} t^{3/2}} e^{-x^2/(4Dt)} = \frac{-2Dt + x^2}{8\pi^{1/2} D^{5/2} t^{5/2}} e^{-x^2/(4Dt)}.
\]

Comparing these expressions, we find that indeed \(\partial c/\partial t = D \partial^2 c/\partial x^2\).
14.7 # 3: From the contour plot, there appears to be a local minimum near (1,1) (enclosed by oval-shaped level curves indicating that as we move away from the point in any direction the values of \( f \) are increasing). Moreover, the shape of the level curves near the origin is characteristic of a saddle point at (0,0).

To verify these guesses, we have \( f(x, y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x = 3x^2 - 3y \) and \( f_y = 3y^2 - 3x \). We have critical points when \( f_x = f_y = 0 \). The first equation \( 3x^2 - 3y = 0 \) gives \( y = x^2 \), and substituting into the second equation gives \( 3(x^2)^2 - 3x = 0 \), hence \( 3x(x^3 - 1) = 0 \), hence \( x = 0 \) or \( x = 1 \). So we have two critical points (0,0) and (1,1).

The second partial derivatives are \( f_{xx} = 6x \), \( f_{yy} = 6y \), \( f_{xy} = -3 \), so \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9 \). Then \( D(0, 0) = 0 - 9 < 0 \) so \( f \) has a saddle point at (0,0); and \( D(1, 1) = 36 - 9 > 0 \), with \( f_{xx}(1,1) = 6 > 0 \), so \( f \) has a local minimum at (1,1).

14.7 # 11: \( f(x, y) = x^3 - 3x + 3xy^2 \Rightarrow f_x = 3x^2 + 3y^2 - 3 \) and \( f_y = 6xy \). Then \( f_x = 0 \iff x^2 + y^2 = 1 \), \( f_y = 0 \iff xy = 0 \).

Thus, at a critical point, either \( x \) or \( y \) is 0, and then the other one is \( \pm 1 \). There are four critical points: \( (\pm 1, 0) \) and \( (0, \pm 1) \).

Next, \( f_{xx} = 6x \), \( f_{xy} = 6y \), \( f_{yy} = 6x \), so \( D = f_{xx}f_{yy} - f_{xy}^2 = 36x^2 - 36y^2 \).

\( D(1, 0) = 36 > 0 \) and \( f_{xx}(1,0) = 6 > 0 \), so (1,0) is a local minimum.

\( D(-1, 0) = 36 > 0 \) and \( f_{xx}(-1,0) = -6 < 0 \), so \((-1,0)\) is a local maximum.

\( D(0, \pm 1) = -36 < 0 \), so (0,1) and (0,-1) are saddle points.

14.7 # 43: We want to minimize the distance from \((4,2,0)\) to \((x,y,z)\), \(d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}\), where \(z^2 = x^2 + y^2\). Instead, it is easier to minimize \(d^2 = f(x,y) = (x-4)^2 + (y-2)^2 + (x^2 + y^2)\). Since \(f_x = 2(x-4) + 2x = 4x - 8\) and \(f_y = 2(y-2) + 2y = 4y - 4\), the only critical point is \((x,y) = (2,1)\). This point must correspond to the minimum distance \((f(x,y) \to \infty \) when \(x\) and/or \(y\) become large). For \(x = 2\) and \(y = 1\), the equation of the cone gives \(z^2 = 5\) or \(z = \pm \sqrt{5}\). Hence the points on the cone closest to \((4,2,0)\) are \((2,1, \pm \sqrt{5})\).

14.7 # 49: Let \((x,y,z)\) be the corner opposite the origin. Since \(z = \frac{1}{3}(6 - x - 2y)\) and the volume is \(xyz\), we want to maximize \(f(x,y) = xyz = \frac{1}{3}xyz(6 - x - 2y)\).

\(f_x = \frac{1}{3}y(6 - 2x - 2y)\), and \(f_y = \frac{1}{3}x(6 - x - 4y)\). Setting \(f_x = f_y = 0\) gives \(2x + 2y = 6\) and \(x + 4y = 6\), hence the only critical point is \((x,y) = (2,1)\), which geometrically must yield a maximum. Thus the volume of the largest box is \(V = f(2,1) = \frac{4}{3}\).