Math 53 Homework 11
Due Wednesday 11/8/17 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 10/30: Green’s theorem
• Read: section 16.4.

Problem 1 below.

Wednesday 11/1: Curl and divergence; vector forms of Green’s theorem
• Read: section 16.5.
• Work: 16.5: (1), (7), 9, 11, (12), 13, (16), (21), 25, (26), (32), 33, (34), 36*, 37.

Problems 2 and 3 below.

* Remark: 16.5 #36 shows that, if a harmonic function $f$ on a domain $D$ is zero everywhere at its boundary curve $C$, then $f = 0$ everywhere in $D$. A consequence is that, if $f_1$ and $f_2$ are two harmonic functions in $D$ such that $f_1 = f_2$ at every point of $C$, then $f_1 = f_2$ everywhere in $D$ (by applying the previous result to the harmonic function $f_1 - f_2$). This uniqueness property has important applications in mathematical analysis.

Friday 11/3: Surface area
• Read: section 16.6.
• Work: 16.6: (3), 13, (18), 23, 24\textsuperscript{1}, (25), (32), (39), 44\textsuperscript{1}, 45, (47).

Problem 1. This problem shows how the value of the integral $I_n = \int_0^{2\pi} \cos^{2n} \theta \, d\theta$ can be determined for all $n$ using Green’s theorem (the conventional method is via integration by parts).

a) Prove that $I_n = \frac{2n - 1}{2n} I_{n-1}$ by writing Green’s theorem for the line integral $\int_C x^{2n-1} \, dy$, where $C$ is the unit circle $x^2 + y^2 = 1$ counterclockwise, and by evaluating each of the integrals separately until it looks like either $I_n$ or $I_{n-1}$.

b) What is the value of $I_0$? Using the result of (a), find expressions for $I_1$, $I_2$, $I_3$ (don’t simplify fractions or calculate products), then give a general formula for $I_n$.

\textsuperscript{1}7th ed: do the 8th ed problems: \# 24: the part of the cylinder $x^2 + z^2 = 9$ that lies above the $xy$-plane and between the planes $y = -4$ and $y = 4$. \# 44: the part of the surface $z = 4 - 2x^2 + y$ that lies above the triangle with vertices (0,0), (1,0), and (1,1).
Problem 2.

a) Let $C$ be the unit circle, oriented counterclockwise, and consider the vector field $\vec{F} = x^2 \hat{i} + xy \hat{j}$. Describe geometrically and/or sketch the vector field $\vec{F}$ (see also 16.2 #32 (b) assigned on HW 10). Which portions of $C$ contribute positively to the flux $\int_C \vec{F} \cdot \hat{n} \, ds$? Which portions contribute negatively?

b) Find the flux of $\vec{F}$ through $C$ by directly evaluating the line integral $\int_C \vec{F} \cdot \hat{n} \, ds$. Explain your answer using (a).

c) Find the flux of $\vec{F}$ through $C$ by using the second vector form (“normal form”) of Green’s theorem.

Problem 3.

Find the flux of the vector field $\vec{F} = \frac{x \hat{i} + y \hat{j}}{x^2 + y^2}$ outwards through:

a) the circle of radius $a$ centered at the origin; do this by calculating the line integral $\int \vec{F} \cdot \hat{n} \, ds$ directly.

b) the circle centered at $(1, 0)$ of radius $b$, for $b \neq 1$. Consider the cases $b > 1$ and $b < 1$ separately, and use Green’s theorem (carefully!). Explain your answers with diagrams.