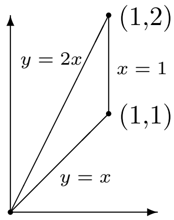
Math 53 – Practice Midterm 2 B – Solutions

1. a)



b) $\int_0^1 \int_{y/2}^{y} dxdy + \int_1^2 \int_{y/2}^{1} dxdy.$

(the first integral corresponds to the bottom half $0 \le y \le 1$, the second integral to the top half $1 \le y \le 2$.)

2. a) $\rho dA = \frac{r \sin\theta}{r^2} r dr d\theta = \sin\theta dr d\theta.$

$$M = \iint_R \rho dA = \int_0^\pi \int_1^3 \sin\theta \; dr d\theta = \int_0^\pi 2\sin\theta d\theta = \big[-2\cos\theta\big]_0^\pi = 4.$$

b) $\bar{x} = \frac{1}{M} \iint_R x\rho dA = \frac{1}{4} \int_0^\pi \int_1^3 r\cos\theta\sin\theta dr d\theta$

The reason why one knows that $\bar{x} = 0$ without computation is that the region **and the density** are symmetric with respect to the y-axis ($\rho(x,y) = \rho(-x,y)$).

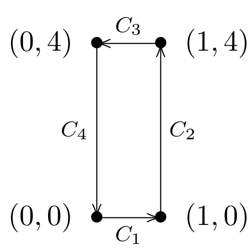
3. a) The parametrization of the circle C is $x = \cos t$, $y = \sin t$, for $0 \le t < 2\pi$; then $dx = -\sin t dt$, $dy = \cos t dt$ and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t)dt + (1 + \cos(\sin t))\cos t dt.$$

b) Let R be the unit disc inside C;

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (Q_x - P_y) dA = \iint_R (0-3) dA = -3 \text{ area}(R) = -3\pi.$$

4. a)



$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dxdy$$
$$= \iint_R (y + \cos x \cos y - \cos x \cos y) dxdy = \iint_R y dxdy$$
$$= \int_0^4 \int_0^1 y dxdy = \int_0^4 y dy = [y^2/2]_0^4 = 8.$$

b) On C_4, $x = 0$, so $\mathbf{F} = -\sin y\, \hat{\mathbf{j}}$, whereas $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$. Hence $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$. Therefore the flux of $\mathbf{F}$ through C_4 equals 0. Thus

$$\int_{C_1+C_2+C_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds - \int_{C_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds\ ;$$

and the total flux through $C_1 + C_2 + C_3$ is equal to the flux through C.

5. Let $u = 2x - y$ and $v = x + y - 1$. The Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$.

Hence $dudv = 3dxdy$ and $dxdy = \frac{1}{3}dudv$, so that

$$\begin{aligned} V &= \iint_{(2x-y)^2+(x+y-1)^2<4} (4 - (2x-y)^2 - (x+y-1)^2)\,dxdy = \iint_{u^2+v^2<4} (4 - u^2 - v^2)\frac{1}{3}dudv \\ &= \int_0^{2\pi}\int_0^2 (4-r^2)\frac{1}{3}rdrd\theta = \int_0^{2\pi}\left[\frac{2}{3}r^2 - \frac{1}{12}r^4\right]_0^2 d\theta = \int_0^{2\pi}\frac{4}{3}d\theta = \frac{8\pi}{3}. \end{aligned}$$

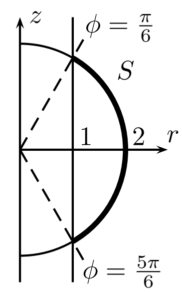
6. a) $P_y = e^x z = Q_x$; $P_z = e^x y = R_x$; and $Q_z = e^x + 2y = R_y$. Thus $\mathbf{F}$ is conservative.

b) We want: $f_x = e^x yz$, $f_y = e^x z + 2yz$, $f_z = e^x y + y^2 + 1$.

Integrating f_x we get: $f(x,y,z) = e^x yz + g(y,z)$. Differentiating with respect to y and comparing, we get: $f_y = e^x z + g_y = e^x z + 2yz$. Thus $g_y = 2yz$, which yields: $g(y,z) = y^2 z + h(z)$.

Plugging back into f, we get: $f(x,y,z) = e^x yz + y^2 z + h(z)$. Differentiating with respect to z and comparing, we get: $f_z = e^x y + y^2 + h'(z) = e^x y + y^2 + 1$, so $h'(z) = 1$. Thus $h(z) = z + c$, and putting everything together we obtain: $f(x,y,z) = e^x yz + y^2 z + z + c$.

7.



a) S is part of the sphere $x^2 + y^2 + z^2 = 4$, so its normal vector points radially outwards, straight away from the origin. So $\hat{\mathbf{n}} = \frac{1}{2}\langle x,y,z\rangle$ (the factor $\frac{1}{2}$ is there because $|\langle x,y,z\rangle| = 2$ at all points of S), and

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \langle y, -x, z\rangle \cdot \frac{\langle x,y,z\rangle}{2} = \frac{z^2}{2}.$$

Using the spherical angles ϕ, θ to parametrize S, $z = 2\cos\phi$, and $dS = 2^2 \sin\phi\, d\phi\, d\theta$, hence the flux is equal to

$$\int_0^{2\pi}\int_{\pi/6}^{5\pi/6} \frac{(2\cos\phi)^2}{2} 4\sin\phi\, d\phi\, d\theta = 16\pi\int_{\pi/6}^{5\pi/6}\cos^2\phi\sin\phi\, d\phi = 16\pi\left[-\frac{\cos^3\phi}{3}\right]_{\pi/6}^{5\pi/6} = 4\sqrt{3}\pi.$$

b) For the cylindrical surface, $\hat{\mathbf{n}} = \pm\langle x, y, 0\rangle$, hence $\mathbf{F}\cdot\hat{\mathbf{n}} = 0$, so the flux is zero.

c) $\operatorname{div}\mathbf{F} = 1$, hence

$$Vol(R) = \iiint_R 1\,dV = \iiint_R \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS + \iint_{\text{Cylinder}} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 4\sqrt{3}\pi.$$

8. a) At any point of S, $z = (x^2 + y^2 + z^2)^2 \geq 0$.

b) $z = \rho\cos\phi$ and $x^2+y^2+z^2 = \rho^2$, so $\rho\cos\phi = \rho^4$. This simplifies to $\cos\phi = \rho^3$, or $\rho = (\cos\phi)^{1/3}$.

c) $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{(\cos\phi)^{1/3}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$

9. The flux is calculated upwards through the graph of $z = f(x,y) = xy$, so

$\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy = \langle -y, -x, 1\rangle\,dx\,dy$. Hence

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{x^2+y^2<1} \langle y, x, z\rangle\cdot\langle -y, -x, 1\rangle\,dx\,dy = \iint_{x^2+y^2<1} (-y^2 - x^2 + xy)\,dx\,dy.$$

Using polar coordinates, we get: $\int_0^{2\pi}\int_0^1 (-r^2 + r^2\cos\theta\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (\cos\theta\sin\theta - 1)r^3 dr\,d\theta = \int_0^{2\pi}\frac{1}{4}(\cos\theta\sin\theta - 1)\,d\theta = \left[\frac{1}{8}\sin^2\theta - \frac{1}{4}\theta\right]_0^{2\pi} = -\pi/2.$