Math 53 Homework 6

Due Wednesday 10/16/13 in section

(The problems in parentheses are for extra practice and optional. Only turn in the underlined problems.)

Monday 10/7 – Max-min problems continued

- **Read:** section 14.7.
- **Work:** Problem 1 below.

Wednesday 10/9 – Lagrange multipliers

- **Read:** section 14.8.
- **Work:** 14.8: (1), 3, (7), 9, (27), 31, (35), 37.[1]
- **Bonus problem** (extra credit, hard): Problem 6 below.

Friday 10/11 – Midterm 1 review

- **Read:** most of chapters 10, 12, 13, 14; practice midterms.
- **Work** (chapter 14 review problems): Problems 2–5 below.

Problem 1. Consider a triangle in the plane, with angles α, β, γ. Assume that the radius of its incircle is equal to 1.

a) By decomposing the triangle into six right triangles having the incenter as a common vertex, express the area A of the triangle in terms of α, β, γ (your answer should be a symmetric expression). Then use your result to show that A can be expressed as a function of the two variables α and β by the formula

$$A = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \tan\frac{\alpha+\beta}{2}.$$

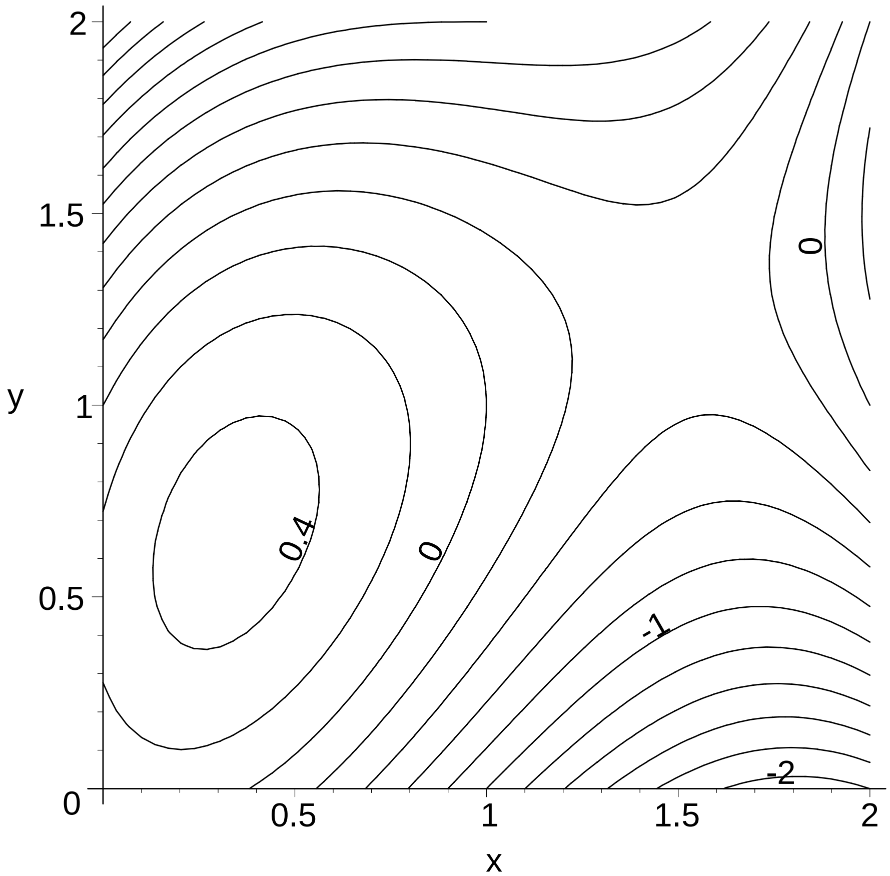
b) What is the set of possible values for α and β? Find all the critical points of the function A in this region.

c) By computing the values of A at the critical points and its behavior on the boundary of the region where it is defined, find the maximum and the minimum of A (justify your answer). Describe the shapes of the triangles corresponding to these two situations.

Problem 2. The figure on the next page is the contour plot of a function of two variables $f(x,y)$, for x and y ranging between 0 to 2 (scale: 1 unit = 5 cm; spacing between contour levels: 0.2).

a) Use the contour plot to determine whether f_x and f_y are > 0, $= 0$, or < 0 at the point $(\frac{3}{2}, \frac{1}{2})$. Same question at the point $(1,1)$?

[1] 6th ed: 14.8: (1), 3, (7), 9, (25), 29, (33), 35.



b) At the point $(\frac{3}{2}, \frac{1}{2})$, find the direction(s) $\hat{u}$ for which the directional derivative of f is zero. (Either sketch a portion of level curve and the direction vector(s), or give the approximate angle(s) from the positive x-axis).

c) Use the contour plot to find two points where $f_x = f_y = 0$, and give their approximate coordinates. What happens to the level curves of f through these points? For each of the two points, describe what happens when you move towards North, South, East, West: does the value of f go up, down, or does it stay exactly the same? What types of critical points are these?

Problem 3.

The function in the previous problem is $f(x, y) = x(x-1)(x-2) + (y-1)(x-y)$.

a) Calculate the actual values of the partial derivatives at $(\frac{3}{2}, \frac{1}{2})$ and $(1, 1)$.

b) At $(\frac{3}{2}, \frac{1}{2})$, find the direction(s) in which the directional derivative of f is zero.

c) Find the points where $f_x = f_y = 0$, and calculate the second partial derivatives at these points. Use this information to confirm the types of the critical points.

(Optional but recommended: compare your answers with those you obtained in Problem 2. How accurate were your estimates from the contour plot?)

Problem 4.

a) Find the direction from $(4, 2, 3)$ in which $g(x, y, z) = x^2+y^2-6z$ decreases fastest.

b) Follow the line in the direction you found in part (a) to estimate, using linear approximation, the location of the point closest to $(4, 2, 3)$ at which $g = 0$. Do not use a calculator. Express your answer using fractions. Next, use a calculator to evaluate g at your point. (The value should be reasonably small.)

(Hint: in the linear approximation, the closest point where $g = 0$ lies on the line through $(4, 2, 3)$ in the direction found in (a). Find a parametric equation for this line, then use linear approximation to estimate the value of g at a point on the line.)

Problem 5. Now we seek an exact answer to Problem 4(b).

a) Use the method of Lagrange multipliers to write down the system of equations satisfied by the point closest to $(4, 2, 3)$ at which $x^2 + y^2 - 6z = 0$. (Hint: it is easier to minimize the square of the distance).

b) Solve the equations you found in (a) to get the exact location of the point. Then use a calculator to evaluate your answers to five decimal places. Compare your answers with the approximation in 4(b). Was each coordinate of the approximate answer in 4(b) within 1/100 of the exact answer?

Problem 6 (extra credit, hard) (only attempt this problem if you're done reviewing for the midterm!)

UC Berkeley has hired a famous architect to design a new building, to be located on a flat, triangular plot of land, with sides of given lengths a_1, a_2, a_3. The building will have the shape of a pyramid, with the base exactly covering the entire plot. The volume of the pyramid is also fixed (in order for the building to accommodate the planned amount of occupants). The architect has decided that the three triangular side faces of the pyramid would be entirely covered in gold. However, to minimize cost, the shape of the pyramid will be chosen so that the sum of the areas of the side faces is the smallest possible one (given the fixed triangular base and fixed volume). The goal of this problem is to find the optimal position of the apex P of the pyramid (the point at the tip).

a) One possibility would be to work in coordinates, expressing the surface area in terms of the coordinates (x, y, z) of the point P, and those of the vertices of the base triangle (which lies in the xy-plane). What is the constraint satisfied by x, y, z? Unfortunately, solving the problem in this way leads to extremely lengthy calculations. (Convince yourself of this! How would you express the area in terms of x, y, z? Don't write the entire formula).

b) The problem is easier to solve if one uses a different set of variables. Denote by Q the point in the base triangle that lies directly beneath P, and let u_1, u_2, u_3 be the distances from Q to the sides of lengths a_1, a_2, a_3 of the triangle respectively. What is the relation between u_1, u_2, u_3? (Hint: decompose the base triangle into 3 smaller triangles with a vertex at Q). What is the total area of the side faces of the pyramid as a function of u_1, u_2, u_3?

c) Use Lagrange multipliers to solve the min/max problem you arrived at in part (b). What can you say about the values of u_1, u_2, u_3 at the solution? Geometrically, what does this say about the point Q?