# Notes on family Floer cohomology

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March 15, 2016

**Abstract**

Notes for a talk at mirror symmetry seminar on Abouzaid's family Floer cohomology [1]. These notes are mostly just my attempts to parse some main points in the paper and give them additional context. These notes may contain errors and certainly contain omissions & simplifications.

## 1 Integral affine structures

The situation for today's talk is the following: we want to investigate mirror symmetry on a compact symplectic manifold $X$ which admits a Lagrangian torus fibration $\pi : X \to Q$ over a base $Q$ with $\pi_2(Q) = 0$. In this talk, we'll assume that the fibration is everywhere smooth, *i.e.*, that the fibers never degenerate to singular tori. In such cases, there's a lot of work to be done figuring out what to do at the singular fibers, but already the case of a nonsingular fibration is interesting.

We have a complete local description of such fibrations by the Arnol'd-Liouville theorem:

**Theorem 1.** *In a neighborhood of each fiber, we can find* action-angle coördinates $(\phi_1, \dots, \phi_n, I_1, \dots, I_n)$ *in which the symplectic form on* $X$ *is given by* $\omega = \sum d\phi_i \wedge dI_i$ *and the fibration* $\pi$ *is just the projection* $(\vec{\phi}, \vec{I}) \mapsto \vec{I}$, *and the coördinates* $I_i$ *are defined up to an ambiguity of* $Aff_{\mathbb{Z}} := GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$.

We can use this description to endow $Q$ with the structure of an integral affine manifold:

**Definition/Proposition.** An *integral affine structure* on a manifold $X$ consists of an atlas whose transition functions are contained in $Aff_{\mathbb{Z}}$. Equivalently, an integral affine structure on $X$ can be specified by the data of a torsion-free flat connection $\nabla$ and a maximal rank $\nabla$-horizontal lattice $T^{\mathbb{Z}} \subset TX$.

By the theorem, the lattice given locally in action-angle coördinates by $\langle dI_1, \dots, dI_n \rangle$ defines a global rank $n$ sublattice of $T^*Q$ and hence an integral affine structure on $Q$.

We can also give a more invariant description of this lattice as follows: letting $F_q$ denote the fiber over $q \in Q$, we have a canonical isomorphism

$$H_1(F_q; \mathbb{R}) \cong T^*_q Q$$

which sends the 1-cycle $\gamma$ to the covector $v \mapsto \int_\gamma \iota_{\tilde{v}} \omega$, where $\tilde{v}$ is a lift of $v \in T_q Q$ to a tangent vector at $X$.

Similarly, we have a dual isomorphism $T_qQ \cong H^1(F_q;\mathbb{R})$. This allows us to define lattices $T_q^{\mathbb{Z}}Q, T_{\mathbb{Z}}^*Q$ in $TQ$ and $T^*Q$ as the images of $H^1(F_q;\mathbb{Z})$ and $H_1(F_q;\mathbb{Z})$, respectively. It turns out that this data almost completely determines the total space $X$:

**Proposition 1.** *$X$ is determined (up to fibrewise symplectomorphism) by the data of $Q$, $T_{\mathbb{Z}}^*Q$, and the class of a certain 2-cocycle $\alpha_X$.*

For the sake of completeness: let $Aff$ be the sheaf of integral affine functions. Then we can construct $[\alpha_X] \in H^2(Q; Aff)$ as the class of the 2-cocycle which measures the global difference in trivializations $X_{P_i} \cong T^*P_i/T_{\mathbb{Z}}^*P_i$.

For most of this talk, we'll take $\alpha_X$ to be zero for simplicity.

## 2 Family Floer cohomology and rigid geometry

The basic philosophy of family Floer cohomology is as follows: pick a distinguished family of lagrangians $\{L_q\} \subset X$. Then, given any Lagrangian $L \subset X$, we not only have the data $CF^*(L_q, L)$, but we also expect these complexes to fit together in some kind of continuous way, for instance as some kind of sheaf.

**Example 1** (Nadler-Zaslow)**.** I can't resist taking a moment to advertise my favorite example of this philosophy: Let $X = T^*M$ be a cotangent bundle. On any small open set $U \subset M$, we can define a certain "standard Lagrangian" $L_U$ (the smoothing of $SS(j_*\mathbb{C}_U)$), and the Nadler-Zaslow theorem says that the data $\{Hom(L, L_U)\}$ can be glued together to form a constructible sheaf on $M$. Nadler proved that standard objects generate the Fukaya category, so this procedure induces a map $\mathcal{F}(X) \to Sh(M)$ which turns out to be an $A_\infty$ quasiequivalence.

In our case, we have a natural set of Lagrangians to test against: the fibers $F_q$ of the fibration $\pi$. Then, from the "family Floer cohomology" philosophy, we might hope that for a fixed lagrangian $L$, the complexes $CF^*(F_q, L)$ form the stalks of some sheaf on some space. Ultimately, we hope for the following:

**Goal.** Produce a space $Y$ and a map $\mathcal{F}(X) \to Coh(Y)$ sending $L$ to a sheaf with stalks given by $CF^*(F_q, L)$.

Actually, we haven't been totally honest so far: objects of the Fukaya category are not actually Lagrangian submanifolds but branes, which is today Lagrangian submanifolds endowed with some additional data, including a local system. So in fact our family of test branes should be parametrized by pairs $(q \in Q, b \in Loc_1(F_q))$; since these are the points at which our proposed sheaf has stalks, we expect that our mirror $Y$ is a space parametrizing the data $(q, b)$. So as a set, $Y$ should be something like $\coprod_{q\in Q} H^1(F_q;\mathbb{C}^*)$.

But we don't just want a discrete set with some vector spaces at each point; we want an honest space, with a coherent sheaf on it. To promote the complexes $CF^*((F_q, b), L)$, which live over the points $(q, b)$ of $Y$, to the stalks of a sheaf, we need some way of *gluing together* these things for nearby Lagrangians. Luckily for us, it turns out that the affine structure we discussed above is exactly what we need to identify local systems on nearby fibers: the affine structure gives us coherent isomorphisms $H^1(F_q) \cong H^1(F_p)$ for $p$ near $q$.

This procedure seems to give us a complex-analytic space $Y$, which is the sort of thing we want. But there's another problem, which we are going to notice once we start doing Floer theory, which is that we don't know whether the Floer differentials we define will actually end up being convergent. To fix this problem, the traditional solution is to work over the *Novikov field*

$$\Lambda := \left\{ \sum_{i=1}^{\infty} a_i t^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \to \infty \right\}$$

instead of $\mathbb{C}$. Which means that really we're not going to be able to define $Y$ as a complex-analytic space at all; instead, it's only going to be a space over the Novikov field. As a set, $Y$ will look like $\coprod_{q \in Q} H^1(F_q; U_\Lambda)$, where $U_\Lambda$ is the units in the subring of integral elements in $\Lambda$, *i.e.*, the subring

$$\left\{ \sum_i a_i t^{\lambda_i} \mid \min(\lambda_i) = 0 \right\}$$

of elements with valuation 0.

What kind of object is $Y$ now? It's certainly no longer a complex-analytic space. In fact, $Y$ is now a *rigid-analytic space.* But really, this isn't so bad. Rigid-analytic spaces, in analogy to the situation in complex analysis or algebraic geometry, are defined to be ringed spaces which are locally equivalent to some standard domain with a standard sheaf of rings on it. The only difference here is that the sheaf of rings, rather than being a sheaf of holomorphic or polynomial functions, is a sheaf of Novikov series which are defined only formally. But we can sitll perform most of the operations which we would have liked to have done in the complex-analytic category.

In particular, we can glue spaces together, which we'll use now to understand charts and local coördinates for $Y$. We define

$$Y := T^{\mathbb{Z}} Q \otimes_{\mathbb{Z}} U_\lambda.$$

Let $\mathrm{val} : Y \to Q$ be the projection to $Q$, and let $P$ be a small neighborhood of $q \in Q$. Then we can lift $P$ to a neighborhood

$$Y_P := \mathrm{val}^{-1}(P) = \coprod_{p \in P} H^1(F_p; U_\Lambda).$$

But if $P$ is sufficiently small, then using the affine structure on $Q$ we can identify $P$ with a neighborhood of the origin in $T_q Q$ and hence embed $Y_P$ as

$$Y_P = \coprod_{p \in P} H^1(F_p; U_\Lambda) \subset H^1(F_q; \Lambda^*).$$

Now we can finally say what the structure sheaf of $Y$ is in coördinates. Its restriction to $Y_P$, for $P$ a small neighborhood of $q$ as above, will be

$$\mathcal{O}_P := \left\{ \sum_{A \in H^1(F_q;\mathbb{Z})} f_A z_q^A \mid f_A \in \Lambda, \sum f_A x^A \in \Lambda \text{ for all } x \in P \right\}.$$

If we had defined $Y$ in an analogous fashion as a complex-analytic space, $\mathcal{O}_P$ would be defined as the series in $z_q$ (i.e., series in a coördinate $z$ which we take to be 0 at $q$) which converge everywhere in $Y_P$; in the non-archimedean field $\Lambda$, we require only the weaker condition of $\mathfrak{m}$-adic convergence.

Thinking of the complex-analytic analogue is helpful also for one final point which will be used in the next section: $\mathcal{O}_P$ could just as easily have been defined as series in $z_p$ which converge everywhere in $P$; making this change precisely entails scaling the coefficients of functions in $\mathcal{O}_P$ in some well-defined fashion. In the analytic case, this scaling means changing the exponent of $t$, which will become important a bit later.

Heuristically, you can think of this in the following way: for $P$ a small neighborhood of $q \in Q$, our coördinates $z_i \in \mathbb{C}$ can be decomposed into two parts: an argument in $S^1$ and a radial part in $\mathbb{R}$ (this corresponds to the decomposition $\Lambda^* = U_\lambda \times \mathbb{R}$), where the $S^1$ part changes the local system and the $\mathbb{R}$ part controls movement on the base. Thus, if we change from a coördinate centered at $q$ to a coördinate centered at $p$, then we adjust for this by a scaling by $\mathbb{R}$ (which corresponds on the Novikov side to a scaling of the exponent of $t$.

# 3 Floer theory

For simplicity, we'll make the technical assumption that $L$ is tautologically unobstructed. For $\phi$ an appropriate Hamiltonian perturbation, we define $CF^*(F_q, \phi(L))$ in the usual way: each intersection point contributes a rank 1 vector space $\delta_x$, and if $\deg(x) = \deg(y) + 1$, the contribution of $\delta_x$ to the differential $d|_{\delta_y}$ is defined by counting elements in a certain moduli space $\mathcal{M}^q(x, y)$.

On the mirror side, what we want is a sheaf of chain complexes; we need to define what this will be on opens $Y_P$ in $P$. Let $P$ be a small neighborhood of some fixed $q \in Q$. Then the underlying graded vector space which our sheaf assigns to $Y_P$ will be

$$\bigoplus_{x \in F_q \cap \phi(L)} \mathcal{O}_P \otimes_{\mathbb{Z}} \delta_x, \tag{1}$$

and our task will be to define a differential making this into a chain complex.

In order to do this, we'll have to choose some *Floer data*, and one can show via the appropriate continuation maps that the choices of Floer data that we've made won't affect the chain complex that we get, up to quasi-equivalence.

Choose a trivialization $\tau : Y_P \cong T^*P/T^*_{\mathbb{Z}}P$. Then the sheet of $\phi(L)$ over $P$ which contains $x$ can be written as $\Gamma_{g_x}$ for $g_x : P \to \mathbb{R}$, which is well-defined up to integral affine functions.

**Definition 1.** The *Floer data* consists of the choices $(\tau_P, \phi, J, \{g_x\})$.

Now we need to say how this Floer data allows us to define a differential on our complex. First, note that if $u$ is a strip whose sides map to $L$ and to $F_q$, then the choices of $g_x$ provide a construction of the class $[\partial u] \in H_1(F_q, \mathbb{Z})$. There should really be a picture included here, but you can probably imagine it: two sides of this cycle correspond to the boundaries $u$, and the other two sides come from the sheets, which we have constructed explicitly via $g_x$ and $g_y$

Thus we can define the restriction of $d$ to $\delta_y$ to be

$$d\,|_{\delta_y} := \bigoplus_x \sum_{u \in \mathcal{M}^q(x,y)} t^{\mathcal{E}(u)} z^{[\partial u]} \otimes d_u. \tag{2}$$

We claim that the series defining this differential is convergent for all $y \in Y_P$. Recall that $y \in Y_P$ corresponds to a pair $p \in P, b \in H^1(F_p; U_\Lambda)$. Note that if we evaluate (1) at $(q, b)$, then we recover precisely the complex $CF^*((F_q, b), \phi(L))$, and moreover if we evaluate (2) at $(q, b)$, then $[\partial u]$ vanishes and we recover exactly the standard Floer differential.

This shows that the series converges at $(q, b)$. What about $(p, b)$ for some $q \neq p \in P$? In this case, we want to put ourselves in the situation we had in the previous paragraph, for which we need to do two things. First, we'll change our complex structure so that $CF^*((F_q, b), \phi(L))$ in the new complex structure is the same as $CF^*((F_p, b), \phi(L))$ in the old complex structure. Second, we need to change coördinates on $Y_P$ from the coördinate $z_q$ (centered at $q$) to the coördinate $z_p$ (centered at $p$). Note that the first change will shift the exponent of $t$ (as we saw in the previous section), and the second change will affect the energy of the map $u$. But the exponent of $t$ is precisely the thing keeping track of the energy of $u$, which should make the following claim reasonable-sounding:

**Claim.** These two changes cancel each other out.

This proves that the series we have defined is convergent and agrees with the Floer differential for all $p \in P$. At this point, you just need to go through all the usual yoga with continuation maps to check that (up to quasi-equivalence) nothing that we've done depends on the Floer data and that all this agrees with restriction maps; *i.e.*, if $P_{12} := P_1 \cap P_2 \neq \emptyset$, then the restriction maps work in the appropriate way and don't depend on our choice of Floer data on $P_1, P_2$, or $P_{12}$.

Finally, we conclude that we have defined a process which takes as input the triple $(Q, T^*_{\mathbb{Z}}Q, \alpha = 0)$ and produced a rigid-analytic mirror $Y$ and a map $\mathcal{F}(X) \to Coh(Y)$, sending the Lagrangian $L$ to the sheaf whose sections on $Y_P$ are the cochain complex defined above. This isn't quite a functor yet because we haven't explained what happens on morphisms, but maybe you can imagine it.

Finally, we'll consider the case $\alpha_X \neq 0$. In this case, the restriction maps among different types of Floer data will only commute up to $\exp(\alpha_X(ijk))$, which is a 2-cocycle representing the class on $H^2(Y, \mathcal{O}^*)$ which corresponds to $\alpha_X$; hence instead of a coherent sheaf on $Y$, we get a sheaf whose restriction maps agree only up to $\alpha_X$, or in other words we get a coherent sheaf on $Y$ twisted by the analytic gerbe $\alpha_X$.

# References

[1] M. Abouzaid. *Family Floer cohomology and mirror symmetry.* arXiv:1404.2659.

[2] M. Kontsevich and Y. Soibelman. *Affine structures and non-archimedean analytic spaces.* arXiv:math/0406564.