I) What we need to define $\mathcal{F}(M)$ as an ungraded $\infty$-cat on $\mathbb{Z}/2$:

**Partial definition:** the Fukaya cat. $\mathcal{F}(M)$ is the $\infty$-cat. with

- objects = \{ Lagrangian submanifolds of $M$ \} (exact, closed, ...)
- $\text{hom}(L_0,L_1) = \text{vector space generated by } L_0 \cap L_1$ (assuming $L_0 \cap L_1$),
- $\mu: \text{hom}(L_0,L_1) \otimes ... \otimes \text{hom}(L_0,L_d) \to \text{hom}(L_0,L_d)$ \\
  given by counts of (d+1)-pointed $\mathbb{J}$-holomorphic discs.

**Problems:**
- what if $L_0,L_1$ not transverse? (e.g. if $L_0 = L_1$?)
- consistency: transversality of moduli spaces?
  - gluing consistency as discs degenerate?
  
  (so $\infty$-relations still hold).

**Def:** Given $L_0,L_1 \subset M$, a *Fukaya datum* := $(H,J)$ where

- $H = t$-dependent Hamiltonian $M \to \mathbb{R}$ ($t \in [0,1]$)
- $J = t$-dependent almost complex structure
- s.t. $L_0 \# \Phi_H(L_0)$ transversely, ($\Phi_H = \text{time } t \text{ flow of } H$)

and s.t. $J$ is regular for $H$-perturbed holom. sheaves, i.e. moduli spaces of solutions of

\[ 3u := \frac{\partial u}{\partial s} + J(t) \left( \frac{\partial u}{\partial t} - X_H(t) \right) = 0 \quad (u: \mathbb{Z} = \mathbb{R} \cup \{0\}) \]

are smooth moduli of the expected dimension.

\* We'll fix a Fukaya datum for every pair of Lagrangians.

**Def:** If $S$ is a (d+1)-pointed disc, a set of *skip-like ends* for $S$ is a collection \{ $E_0 \}$, $S$ boundary marked point, of holom. maps $E: \mathbb{Z}^+ \to S$ with $\mathbb{R}^+ \times \{0\} \to S$ for the marked pt $1, ..., d$ (**"positive"** puncture = input) and $E: \mathbb{Z} = \mathbb{R} \cup \{0\} \to S$ similarly for marked pt 0 (**"negative"** puncture = output).
If $S$ is thought of as punctured at $5$'s, a set of Lagrangian labels for $S$ is an assignment of a Lag. subvar. $L_i$ to each component $C_i$ of $S$.

Note: we then have a chosen Floer datum at each end, by the above.

* Given a pointed disk $S$ with strip-like ends & Lag. labels, define a perturbation datum $(K,J)$ to consist of:
  
  \[
  \begin{cases}
  & \text{a 1-form } K \in \mathcal{L}^1(S, H) \\
  & \text{of space of Hamiltonians} \\
  & \text{a domain-dependent } J \in \mathcal{C}^\infty(S, J).
  \end{cases}
  \]

  s.t.:
  
  \[
  \begin{cases}
  & \text{in each strip-like end}, \ (K,J)|_{\xi_5} = (H dt, J) \text{ the given Floer data for pair } (L_i, L_i^-), \ \\
  & K(\xi) |_{L_i} = 0 \text{ for } \xi \in TC_i: \text{ a vector tangent to } \partial \text{-compt. } C_i \text{ labelled by } L_i.
  \end{cases}
  \]

Issue: need to choose these perturbation data consistently, i.e. as $S$ degenerates to a pair of discs, need perturbation in the newly formed strip-like end to agree with previously chosen Floer data for the appropriate Lagrangians.

ie: if $L_3$ degenerates to $L_0$, then in this region $(K,J)$ needs to agree with Floer data for $(L_0, L_2)$.

Do this inductively on associahedra: $R^4 = \begin{array}{c}
\end{array}$

\[
\begin{array}{c}
\end{array}
\]
Then \( \mathbb{R}^4 = \mathcal{C} \)

Choose gluing chart
\[
\varphi: \mathbb{R}^3 \times \mathbb{R}^4 \times (0,1) \to \mathbb{R}^3 \quad ...
\]

st. 1) ship-like ends agree across gluing:

\[
\begin{array}{c}
\circ \quad \circ \\
\rightarrow \\
\circ 
\end{array}
\]

the ship-like ends of glued disc agree with them of compact discs for small values of gluing parameter.

2) similarly for perturbation data

(However, only require pert. data to strictly agree on “thin part” i.e. ship-like ends + glued pieces, and to converge to pert. data of pieces as gluing parameter \( \to 0 \) on thick parts).

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II) gradings: we know (assuming transversely) that for fixed domain

\[
\dim M = \text{ind } D_3 = \text{some topological quantity} \Rightarrow \text{Maslov index.}
\]

To get a \( \mathbb{Z} \)-grading on the Fukaya cat, need to assume: \( 2c_1(M) = 0 \).

We will only consider graded Lagrangian subspaces.

Since \( 2c_1(M) = 0 \), \( \exists \) global trivialization of \( (\Lambda^2 TM)^{\otimes 2} \), i.e. a nonvanishing section \( \Omega \) of it, and we can consider the "phase-squared" map \( \phi(L) = (\text{vol}_L)^2/\Omega : L \to S^4 \).

In fact, \( \phi \) is a map defined on the Lagrangian Grassmannian bundle

\[
\begin{array}{c}
LGr(M) \\
\downarrow \phi \\
M \end{array}
\]

(Nota: in \( \mathbb{C}^n \), Lagr. planes = image of \( A: \mathbb{R}^n \to \mathbb{C}^n, \quad A \in U(n)/O(n) \) then \( \phi \) is essentially \( (\det A)^2 \))
The Narain class of \( L \) is \( \mu \in \mathbb{H}^1(L, \mathbb{Z}) \) defined by \( \phi(L) \in \mathbb{H}(L, S^1) \approx \mathbb{H}^1(L, \mathbb{Z}) \).

If \( \mu = 0 \) then \( L \) lifts to a section of \( \widetilde{LGr}(M) \), and conversely.

A graded Lagrangian \( \tilde{L} \) is a pair \( (L, l) \) where \( L \subset M \) is Lagrangian (necessarily \( \mu = 0 \)) and \( l \) is a lift to \( \widetilde{LGr}(M) \).

If \( \tilde{L}_0, \tilde{L}_1 \) are graded Lagrangians, and \( x \in L_0 \cap L_1 \), transverse \( \mathfrak{h}_L \), then \( \exists! \gamma: \{0,1\} \rightarrow \widetilde{LGr}(M) \) connecting the lifts of \( T_x\tilde{L}_0 \) and \( T_x\tilde{L}_1 \) up to homotopy.

The index \( i(\tilde{L}_0, \tilde{L}_1, x) \) is related to the winding number of \( \phi(\gamma) \) (see next time). In \( \dim 1 \) it's \( \int \text{winding} \).

Index formula: \( \text{Ind}((D_S)) = i(x_0) - \sum_{x \neq x_0} i(x) \).

Example: \( i(q) - i(p) = 1 \).

\[ \phi(L_0) = 0 \]
\[ 0 < \phi(L_1) < 1 \]
\[ 1 < \phi(L_2) < 2 \]
\[ \text{Ind} = 2 - (1 + 1) = 0. \]