Khovanov homology from Fukaya categories of Hilbert schemes

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Outline

1. Khovanov homology
2. Fukaya categories of Hilbert schemes
3. Formality of the Fukaya category
In 1999, Khovanov introduced a (bigraded) homology group $Kh$ associated to oriented links in $\mathbb{R}^3$.

- The euler characteristic of $Kh$ is the Jones polynomial.
- $Kh$ is defined using a projection to $\mathbb{R}^2$, and skein relations associated to crossing changes.
- Kronheimer and Mrowka have recently proved that $Kh$ detects the unknot.

**Question**

*What is the geometric meaning of $Kh$?*

- In 2005, Seidel and Smith defined *symplectic Khovanov homology*, a singly-graded analogue of Khovanov homology.
- Motivated by homological mirror symmetry, Cautis and Kamnitzer gave a construction of Khovanov homology in terms of derived categories of coherent sheaves (also Thomas).
- Alternative proposal, using gauge theory, due to Witten.
Links and braids

Construct Khovanov homology using \textit{braid closure}:

Crossingless matchings are objects of a category; braids define bimodules.

\[ \text{Kh}(K) = \text{Kh}_{\phi}(C_+, C_-). \]
Arc algebra (Khovanov 2001)

For each integer $0 \leq n$, define the arc algebra $H_n$ to be a graded linear category with

- **Objects**: crossingless matchings on the set $\{1, \ldots, 2n\} \subset \mathbb{R}$ contained in the upper half-plane.

- **Morphisms**: $H_n(C_0, C_1) = (H^*(S^2))^\otimes k [k - n]$ where $k$ is the number of components of $C_0 \cup \overline{C}_1$.

Multiplication is defined *diagramatically* starting with the cup product on $H^*(S^2)$ and the trace $H^*(S^2) \to \mathbb{Z}$. 
Composition of braids and tensor product of bimodules

Braids form a group, with generators the elementary braids:

The category of bimodules over $H_n$ has a natural tensor product

$$P \otimes_{H_n} Q(C_0, C_1) \equiv \bigoplus_{C, C' \in \text{Ob}(H_n)} P(C_0, C) \otimes Q(C', C_1)/ \sim .$$

Assuming that we have a bimodule $\text{Kh}_s$ for each elementary braid,

$$\text{Kh}_\phi \equiv \text{Kh}_{s_0} \otimes_{H_n} \text{Kh}_{s_1} \otimes_{H_n} \cdots \otimes_{H_n} \text{Kh}_{s_k} \quad \text{if } \phi = s_0s_1 \cdots s_k.$$
Cap and Cup bimodules

There is a canonical identification between matchings of $2n - 2$ points, and matching of $2n$ points in which successive integers \{i, i + 1\} are connected by an arc.

For each $i \in \{1, \ldots, 2n - 1\}$, can extend this assignment to a functor

$$\cap_i : H_{n-1} \rightarrow H_n$$

and hence an $H_{n-1} - H_n$-bimodule, and an $H_n - H_{n-1}$ bimodule:

$$\cap_i(C_0, C_1) = H_n(\cap_i(C_0), C_1)$$
$$\cup_i(C_0, C_1) = H_n(C_0, \cap_i(C_1))$$
Elementary bimodules from adjunctions

Proposition (Khovanov 2001)

The bimodules $\cap_i$ and $\cup_i$ are bi-adjoint.
In particular, we have maps of bimodules

$$
\begin{align*}
\epsilon &: \cup_i \otimes H_{n-1} \cap_i \to \Delta H_n \\
\eta &: \Delta H_n \to \cup_i \otimes H_{n-1} \cap_i
\end{align*}
$$

Khovanov defines $\text{Kh}_{\sigma_i} = \text{Cone}(\eta)$ and $\text{Kh}_{\sigma_i^{-1}} = \text{Cone}(\epsilon)$.

![Diagram](image)

Remark

Khovanov homology is determined by the categories $H_n$ and the bimodules $\cap_i$. 
Overview

Seidel and Smith defined \textit{symplectic Khovanov homology} $\mathcal{K}_h(K)$:

1. Construct symplectic manifolds $Y_{2n}$.
2. Associate to each matching $C$ a Lagrangian $L_C$ in $Y_{2n}$.
3. Define the \textit{symplectic arc algebra} $\mathcal{H}_n$ to be the subcategory of the Fukaya category with these objects.
4. Use \textit{Lagrangian correspondences} in $Y_{2n-2} \times Y_{2n}$ to define $\wedge_i$ bimodules. Obtain a bimodule $\mathcal{K}_h\phi$ for each braid $\phi$.
5. If $K$ is obtained from matchings $C_0$ and $C_1$ and a braid $\phi$, define

$$\mathcal{K}_h(K) \equiv \mathcal{K}_h\phi(L_{C_0}, L_{C_1}).$$

In the original construction, $Y_{2n}$ is a \textit{nilpotent slice} of type $(n, n)$ in $\mathfrak{sl}_{2n}$. 
Manolescu observed that these spaces are also open subschemes of Hilbert schemes of points on complex surfaces. Given such a polynomial $p$ with no multiple roots, consider the $A_{2n}$ Milnor fibre

$$\{ u^2 - v^2 = p(z) \} \subset \mathbb{C}^3$$

The projection to $z$ has critical values exactly at the roots of $p$. The projection is quadratic in local coordinates near the critical points. This is an example of a *Lefschetz fibration*. 
Matching spheres

A construction of Donaldson assigns to each embedded arc in the plane connecting critical points a *matching sphere*, which is a Lagrangian $S^2 \subset A_{2n}$:

A crossingless matching defines a Lagrangian submanifold of $(A_{2n})^n$ which is disjoint from the diagonal. This descends to the symmetric product $\text{Sym}^n(A_{2n})$. 
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The Hilbert scheme

The Hilbert scheme of points $A_{2n}^{[n]}$ is the “moduli space” of 0-dimensional schemes of length $n$ on $A_{2n}$. Given a scheme $Z$ of length $n$, projection to the $z$ coordinate gives a scheme in $\mathbb{C}$.

Definition (Seidel-Smith, Manolescu)

$Y_{2n} \subset A_{2n}^{[n]}$ consists of schemes whose projection has length $n$.

Proposition (Seidel-Smith)

Every crossingless matching $C$ defines a Lagrangian $L_C$ in $Y_{2n}$ which is diffeomorphic to $(S^2)^n$, and is canonical up to isotopy.
A quick reminder about Fukaya categories

$Y_{2n}$ is an affine variety with vanishing first chern class. These properties suffice to define a *Fukaya category* $\mathcal{F}(Y_{2n})$ which is a $\mathbb{Z}$ graded, linear, $A_\infty$ category with the following properties

1. Every simply connected Lagrangian in $Y_{2n}$ defines an object of $\mathcal{F}(Y_{2n})$ which is uniquely defined up to shift.
2. The cohomology of the group of morphisms between $L_0$ and $L_1$ is the *Lagrangian Floer homology* $HF^*(L_0, L_1)$.

\[
\]

**Definition**

The symplectic arc algebra $\mathcal{H}_n$ is the subcategory of $\mathcal{F}(Y_{2n})$ with objects the Lagrangians constructed from crossingless matchings.

**Remark**

Morphism spaces agree, e.g.

\[
H_{2n}(C, C) = H^*(S^2) \otimes^n = H^*((S^2)^n) = HF^*(L_C, L_C).
\]
For \( i \in \{1, \ldots, 2n - 1\} \), we have an inclusion \( A_{2n-2} \subset A_{2n} \):

\[
\Gamma_i \cong Y_{2n-2} \times S^2 \subset Y_{2n-2} \times Y_{2n}.
\]

Wehrheim-Woodward and Ma’u assign to compact Lagrangians in products of symplectic manifolds a bimodule over Fukaya categories using *quilted Floer theory*. 
Cohomological equivalence

One can check that the MWW construction works in this setting despite the non-compactness of $\Gamma_i$, yielding:

$$\forall i \in \mathcal{H}_{n-1} - \operatorname{bimod} \mathcal{H}_n$$

**Theorem (Rezazadegan)**

There are cohomological equivalences of categories $H^*(\mathcal{H}_n) \cong H_n$ and of bimodules $H^*(\forall_i) \cong \cap_i$.

**Corollary**

For any knot $K$, there is a spectral sequence

$$\operatorname{Kh}(K) \Rightarrow \mathcal{K}h(K)$$

**Remark**

Werheim and Woodward have laid the foundations for quilted Floer theory over arbitrary rings, but the full detail of the construction of signs is not yet in the literature.
Formality (Joint work with Ivan Smith)

Proposition (Joyce-Waldron, A-Smith)

The categories $\mathcal{H}_n$ are formal, i.e. we have an equivalence of $A_\infty$ categories

$$\mathcal{H}_n \simeq H_n.$$  

Joyce-Waldron prove a similar result for compact complex Lagrangians in holomorphic symplectic manifolds. Our approach uses partial compactifications of $Y_{2n}$, and also yields:

Theorem (A-Smith)

There is an equivalence of $A_\infty$ bimodules

$$\bigoplus_i \simeq \bigcap_i.$$  

In particular, for any knot $K$, there is an isomorphism

$$\text{Kh}(K) = \mathcal{Kh}(K).$$
Seidel’s formality Lemma

Given an $A_\infty$ algebra $\mathcal{A}$, the Hochschild cohomology $HH^*(\mathcal{A}, \mathcal{A})$ is the cohomology of (a completion of)

$$
\cdots \leftarrow \text{Hom}(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}, \mathcal{A}) \leftarrow \text{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A}) \leftarrow \text{Hom}(\mathcal{A}, \mathcal{A}) \leftarrow \mathcal{A}.
$$

The projection to the first term defines a restriction map

$$HH^*(\mathcal{A}, \mathcal{A}) \to H^*(\mathcal{A}).$$

If the differential on $\mathcal{A}$ vanishes (minimal), then the kernel admits a map to $\text{Hom}(\mathcal{A}, \mathcal{A})$.

Lemma (Seidel) 

If $\mathcal{A}$ is minimal and defined over a field of characteristic 0, then $\mathcal{A}$ is formal if and only if there is a pure class $[b] \in HH^1(\mathcal{A}, \mathcal{A})$, i.e. the restriction to $\mathcal{A}$ vanishes, the associated endomorphism of $H^*(\mathcal{A})$ is the Euler vector field

$$[a] \mapsto \deg([a]) [a].$$
Moduli spaces of holomorphic discs in compactifications

Consider a symplectic manifold $\overline{M}$, with a divisor $D$ representing both $c_1(\overline{M})$ and the symplectic form such that $M$ is the complement of $D$ and $\overline{M}$ is “convex” at infinity. Consider discs with boundary on Lagrangians in $M$, intersection number 1 with $D$. Pick a conormal section, and restrict to maps for which the value lies in $\mathbb{R}^+$. 

![Diagram of moduli space of holomorphic discs](image)
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Consider discs with boundary on Lagrangians in $M$, intersection number 1 with $D$. Pick a conormal section, and restrict to maps for which the value lies in $\mathbb{R}^+$. 

![Diagram of discs and Lagrangians](image-url)
Hochschild cohomology classes from partial compactifications

The moduli of spheres encodes a Gromov-Witten invariant:

$$\sum_{c_1(\beta)=1} GW_{0,1}^\beta \in H^2(\overline{M}).$$

**Lemma**

*If the restriction of the above class to M vanishes, then the choice of a bounding cycle defines a class in*

$$\text{HH}^1(\mathcal{F}(M), \mathcal{F}(M))$$

**Remark**

*This approach to proving formality is related to work of Seidel and Solomon on the mirror of $\mathbb{C}^*$-actions on categories of coherent sheaves.*
Partial compactifications of Milnor fibres

Let $\overline{A}_{2n}$ denote the partial compactification of $A_{2n}$

$$\{U^2 - V^2 = W^2 p(z)\} \subset \mathbb{CP}^2 \times \mathbb{C}$$

$D$ consists of the lines $W = 0$ and $U = \pm V$. 

![Diagram](image-url)
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Partial compactifications of Hilbert scheme

Define $\overline{Y}_{2n}$ to be the open subset of $\overline{A}_{2n}^{[n]}$ consisting of schemes whose projection to $\text{Sym}_n(\mathbb{C})$ has length $n$. The divisor at infinity is

$$D_{2n} = \{ Z | \text{Supp}(Z) \cap D \neq \emptyset \}.$$

Using the projection to $\text{Sym}_n(\mathbb{C})$, check that $\overline{Y}_{2n}$ satisfies the desired properties. Let $[b_n]$ denote the associated class in $\text{HH}^1(\mathcal{H}_n, \mathcal{H}_n)$.

**Lemma**

$[b_n]$ is pure.

The proof reduces to computing the moduli space of discs in $\overline{A}_{2n}$ with boundary on a matching sphere.
Formality for bimodules

Assume $\mathcal{A}$ and $\mathcal{B}$ are minimal $A_\infty$ categories, and $\mathcal{P}$ a minimal bimodule. We have natural maps

$$\text{HH}^*(\mathcal{A}, \mathcal{A}) \xrightarrow{\iota_A} \text{H}^*(\text{End}(\mathcal{P})) \xleftarrow{\iota_B} \text{HH}^*(\mathcal{B}, \mathcal{B}).$$

**Lemma**

$\mathcal{P}$ is a formal bimodule if and only if there are cycles

$$b_A \in CC^1(\mathcal{A}, \mathcal{A}), \ b_B \in CC^1(\mathcal{B}, \mathcal{B}), \text{ and } c_\mathcal{P} \in \text{End}^0(\mathcal{P})$$

such that the following properties hold:

1. $b_A$ and $b_B$ are pure
2. The differential of $c_\mathcal{P}$ is $\iota_A b_A - \iota_B b_B$
3. $c_\mathcal{P}$ is pure, i.e. it induces the Euler vector field

$$\mathcal{P}(X, Y) \rightarrow \mathcal{P}(X, Y)$$

for every pair of objects $X$ and $Y$. 
Formality and quilts

For $i = \{0, 1\}$, let $\overline{M}_i$ be symplectic manifolds with divisors $D_i$. Let $\overline{\Gamma}$ be a Lagrangian in $\overline{M}_0 \times \overline{M}_1$. For simplicity, assume

- $H^1(M_i) = H^1(\Gamma) = 0$
- $\Gamma \to M_0$ is a submersion, and $\Gamma \to M_1$ is an embedding.

Under these assumptions, $\Gamma$ defines a functor from $\mathcal{F}(M_0)$ to $\mathcal{F}(M_1)$. Let $L_1$ be the image of a Lagrangian $L_0$ under $\Gamma$, and assume that the $HH^*(HF^*(L_i, L_i))$ classes defined by $D_0$ and $D_1$ are both pure.

**Lemma**

If $\overline{\Gamma} \cap (D_0 \times \overline{M}_1) = \overline{\Gamma} \cap (\overline{M}_0 \times D_1)$, then the $HF^*(L_0, L_0)$-$HF^*(L_1, L_1)$ bimodule defined by $\Gamma$ is formal.