Homological Mirror Symmetry for toric manifolds
Joint work with Fukaya, Oh, Ohta, Ono.

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Outline

1. Toric symplectic manifolds
2. Fukaya categories and generation
3. Floer theory of fibres
Overview

The problem of classifying symplectic manifolds is completely open. It is significantly simplified in the presence of a group action:

Theorem (Delzant)

There is a bijective correspondence between

- compact symplectic manifolds of dimension $n$ with a Hamiltonian $\mathbb{T}^n$ action (up to equivariant symplectomorphism)
- polytopes in $\mathbb{R}^n$ (up to translation and the action of $GL(n, \mathbb{Z})$)

satisfying the Delzant condition:

The normal vectors to the adjacent facets to each corner form a basis for $\mathbb{Z}^n$.

The standard examples are complex projective spaces $\mathbb{C}P^n$. Non-examples include more general Grassmannians, or complete intersections of sufficiently high degree. These are still amenable to toric methods via toric degenerations.
Dimension 1

On $\mathbb{R} \times S^1$, consider the symplectic form $\pi du \wedge dt$

The inverse image of an interval of length $\ell$ is an annulus of area $\pi \ell$. By collapsing each boundary circle to a point, we obtain a sphere of area $\pi \ell$. The symplectic form and the circle action extend smoothly:

$$u(x, y) = |x|^2 + |y|^2$$
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$$u(x, y) = |x|^2 + |y|^2$$
On $\mathbb{R}^2 \times \mathbb{T}^2$, consider the symplectic form $\pi du_1 \wedge dt_1 + \pi du_2 \wedge dt_2$. This form vanishes on $(u_1, u_2) \times \mathbb{T}^2$ which is therefore a Lagrangian submanifold.

$\text{GL}(2, \mathbb{Z})$ acts by diffeomorphisms on $\mathbb{T}^2$. If we simultaneously act by the transpose matrix on $\mathbb{R}^2$, the symplectic form is preserved.

Consider a half-plane

$$\lambda(u_1, u_2) \leq \mu$$

with $\lambda$ primitive integral. $\lambda$ corresponds to a circle subgroup $S^1 \subset T^2$.

Collapse the circle direction in the inverse image of the boundary.
Choose a set of half-planes whose common intersection is compact.

If the two normals at a corner span \( \mathbb{Z}^2 \), we can change coordinates so that the functions are \( u_1 \) and \( u_2 \). We have a model projection

\[
\mathbb{C}^2 \to \mathbb{R}^2 \\
(z, w) \mapsto (|z|^2, |w|^2).
\]
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Examples

$S^2(1) \times S^2(1)$

$S^2(1) \times S^2(2)$

$\mathbb{CP}^2(1)$

$\text{Bl}_\epsilon(\mathbb{CP}^2)$

$\text{Bl}_{1/3}(\mathbb{CP}^2)$
Lagrangian submanifolds

Question

What are the Lagrangian submanifolds of a toric manifold?

This question is too hard to answer. Simpler questions:

1. What is the cohomology?
2. What are the intersection properties?
3. What is the Floer theory of such Lagrangians?

We will essentially completely answer the third question, by proving homological mirror symmetry.

Theorem (A-Fukaya-Oh-Ohta-Ono)

The Fukaya category of a compact toric manifold is generated by local systems on torus fibres, and is equivalent to the category of matrix factorisation of the mirror potential.
Coefficient rings

In this talk, we will consider Novikov rings over the complex numbers $\mathbb{C}$.

\[ \Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \to +\infty} \lambda_i = +\infty \right\} \]

\[ \Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid 0 \leq \lambda_i \right\} \subset \Lambda \]

\[ \Lambda_+ = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid 0 < \lambda_i \right\} \subset \Lambda_0. \]

**Question**

*What additional information can be extracted by working over $\mathbb{Q}$?*
Fukaya categories I

Let $M$ be a symplectic manifold of dimension $n$.

1. The objects of the “pre”-Fukaya category are pairs $(L, b)$, with $L$ a connected spin Lagrangian, and

$$b \in H^1(L, \mathbb{C}^* + \Lambda_+) \oplus \bigoplus_{1<k} H^{2k+1}(L, \Lambda_0).$$

The classes in $H^1$ are obtained by exponentiating $H^1(L, \Lambda_0)$, and correspond to local systems with coefficients in $\Lambda_0$.

2. Morphisms are given by the $\Lambda$ vector spaces

$$CF^*((L_0, b_0), (L_1, b_1)) = \bigoplus_{x \in L_0 \cap L_1'} \langle x \rangle \cdot \Lambda$$

If $L_0 = L_1$, we can instead use $H^*(L_0, \Lambda)$.

3. There are operations $\mu^d$ on morphisms in the Fukaya category, starting with

$$\mu^0 \in \bigoplus_{0 \leq k} H^{2k}(L, \Lambda_0)$$
Fukaya categories II

Definition

The objects of the Fukaya category $\mathcal{F}(M)$ are the pairs $(L, b)$ for which $\mu^0 \in H^0(L, \Lambda_0)$. These are called weakly unobstructed.

For such Lagrangians, $\mu^0$ is a multiple of the identity, so we obtain a function

$$\text{Po}(L, b) = \sum_{\beta \in \pi_2(M, L)} n_\beta T^{\langle \beta, \omega \rangle \langle \partial \beta, b \rangle}.$$

For each scalar $\lambda$, we obtain a subcategory

$$\mathcal{F}_\lambda(M) \subset \mathcal{F}(M).$$

The next term $\mu^1$ is a differential.

The map $\mu^2$ defines a multiplication

$$CF^*(U, V) \otimes CF^*(V, W) \to CF^*(U, W)$$
Generators

If $\mathcal{L}$ is a subcategory of the Fukaya category (with objects $(L, b)$), we have a functor

$$\mathcal{F}(M) \to \text{mod } -\mathcal{L}$$

$$U \mapsto Y_U \equiv \bigoplus_{(L, b) \in \text{Ob}(\mathcal{L})} CF^*(U, (L, b)).$$

**Definition**

$\mathcal{L}$ generates $\mathcal{F}(M)$ if this functor is a fully faithful embedding:

$$CF^*(U, V) \simeq \text{Hom}_\mathcal{L}(Y_U, Y_V).$$

**Remark**

Let $\mathcal{L}_\lambda = \mathcal{L} \cap \mathcal{F}_\lambda(M)$. If $\mathcal{L}$ generates, then, for each $\lambda$ we have an embedding

$$\mathcal{F}_\lambda(M) \to \text{mod } -\mathcal{L}_\lambda$$
Fibres are weakly unobstructed

$\mathcal{M}_P$ is a compact toric manifold corresponding to a polytope $P$. Let $L_u$ denote the Lagrangian torus fibre at $u$.

**Proposition (FOOO)**

*If $L$ is a torus fibre in a compact toric manifold, and $b \in H^1(L)$, then $(L, b)$ is weakly unobstructed.*

**Sketch of proof.**

The degree $k$ part of $\mu^0(L, b) \in H^*(L, \Lambda_0)$ is controlled by the moduli space of stable discs whose *Maslov index* is $2 - k$. The inverse image of $\partial \mathcal{P}$ represents twice the Maslov class, so the degree $k$ part is controlled by stable discs whose intersection with the inverse image of the boundary is $k$.

Positivity of intersection implies that all holomorphic discs have Maslov index greater than 2, so they only contribute to the degree 0 part.
Let \( \Lambda^* = \Lambda - \{0\} \), and consider the valuation
\[
\nu: \Lambda^* \rightarrow \mathbb{R}
\]
\[
\nu \left( \sum a_i T^{\lambda_i} \right) = \min_{a_i \neq 0} (\lambda_i)
\]

The inverse image of a polygon \( P \subset \mathbb{R}^n \) under the valuations on each component is a domain
\[
\Omega_P \subset (\Lambda^*)^n \cong H^1(\mathbb{T}^n, \Lambda^*).
\]

This is a *rigid analytic scheme* which parametrises pairs \((L_u, b)\). The function \( P_0 \) is regular in these coordinates:
\[
P_0(L_u, b) = \sum_{\beta \in \pi_2(M_P, L_u)} n_\beta T^{\langle \beta, \omega \rangle} \langle \partial \beta, b \rangle
\]

**Proposition**

\((L_u, b)\) is a non-zero object of the Fukaya category if and only if it is a critical point of \( P_0 \).
Examples

- $2T^{1/2}$
- $-2T^{1/2}$

Diagrams of toric symplectic manifolds and Fukaya categories, including examples of $2T^{1/2}$ and $-2T^{1/2}$.
Homological mirror symmetry: I

The category of matrix factorisations of Po is a deformation of the category of sheaves on $\Omega_P$. It has an explicit set of generators “supported” on the critical points of Po.

**Theorem**

*The derived Fukaya category of $M_P$ is equivalent to the category of matrix factorisations of Po.*

The first step of the proof is to express the higher products on the Floer cohomology of toric fibres in terms of Po:

1. The first derivative of Po determines the differential $\mu^1$
2. If the homology with respect to $\mu^1$ vanishes, the Hessian determines the product $\mu^2$
3. If the critical point is degenerate, third derivatives determine $\mu^3$
4. and so on.

This yields an equivalence between generators of the category of matrix factorisations and Lagrangian torus fibres.
The next step is to prove that the Lagrangian torus fibres generate the Fukaya category. There is a general criterion for this:

**Theorem**

*A subcategory $\mathcal{L}$ generates the Fukaya category if the identity in $QH^0(M)$ lies in the image of the map*

$$HH_*(\mathcal{L}, \mathcal{L}) \to QH^0(M).$$

For toric manifolds, let $\mathcal{L}$ be the category consisting of fibres.

$$HH_*(\mathcal{L}, \mathcal{L}) \to HH_*(\text{Mat}(Po), \text{Mat}(Po)) \to QH^*(M_P) \to \text{Jac}(Po)$$