Recall: \( L_0 = \nu, L_1, L_2 = \nu(L_1) \subseteq M \)

Continued: \( c \in CF(L_1, L_2) \) \( \mu^1(c) = 0 \)
\( k : CF(L_0, L_1) \rightarrow CF(L_0, L_2) \) \( \mu^2(c, \cdot) + k(\mu^1(\cdot)) + \mu^1(k(\cdot)) = 0 \).

Now we wish to upgrade this to a contract on the associated homology module. Let \( Q_0 \) be a family of fibrations interpolating this.

Continued: \( d : CF(Q, L_1) \rightarrow CF(Q, L_2) \)
\( \ell : CF(L_0, L_1) \otimes CF(Q, L_0) \rightarrow CF(Q, L_2) \)
\( \mu^1(d(\cdot)) + d(\mu^1(\cdot)) = 0 \)
\( d(\mu^2(\cdot, \cdot)) + \mu^1(\ell(\cdot, \cdot)) + \ell(\mu^1(\cdot, \cdot)) = 0 \)

This means that \( D = (CF(L_0, L_1) \otimes CF(Q, L_0)) \oplus CF(Q, L_1) \oplus CF(Q, L_2) \)

is a 

```
\begin{tikzcd}
\mu^2 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\mu^1 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\mu^1 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\end{tikzcd}
```

is a 

```
\begin{tikzcd}
\mu^1 \mu^2 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\mu^1 \mu^2 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\mu^1 \mu^2 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\end{tikzcd}
```

This (Serre): The complex \((D, D)\) is acyclic.

Contrasting \( d \) and \( \ell \):

* \( d \) counts (index 0) sections of:

```
\begin{tikzcd}
L_2 \otimes \nu (x_2) \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
L_1 \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
Q \arrow[r, shift right=0.5ex, shift left=0.5ex, out=90, in=90]
\end{tikzcd}
```

* \( \ell \) counts (index -1) sections of family of fibrations interpolating this.

Show \( \mu^1 d + d \mu^1 = 0 \) by considering \( \ell \) of \( \nu - d \mu^1 \) as an index 1 section (= local index 1 sheaf).

* \( \ell \) counts index -1 sections of family of fibrations interpolating this.
\[ C^0(E^{l,0}/S^{l,0}) = 0 \] (vanishing \( H_{nm} \))

\[ M^0(E^{l,0}/S^{l,0}) = \text{1-dim! manifold with boundary} \]
- \( M^0(E^{l,0}/S^{l,0}) \)
- \( M^0(E^{l,1}/S^{l,1}) \)
- \( \bigsqcup_{e(0,1)} (M^{-1}(E^{l,r}/S^{l,r}) \times \text{shaps}) \)

\[ \Rightarrow \text{properties follow.} \]

* Moreover: if \((\tilde{d}, \tilde{e})\) arise from different construction choices, then can still show \((\tilde{d}, \tilde{e})\) is homotopic to \((d, e)\).

This applies in particular to:
\[
\begin{align*}
\tilde{d} &= \mu^2(c, \cdot), \\
\tilde{e} &= \mu^2(k(\cdot), \cdot) + \mu^3(c, \cdot, \cdot)
\end{align*}
\]

Since \( \tilde{d} = \mu^2(c, \cdot) \) counts section of \( \Gamma_{TV} \) - degenerate limit of contr. of \( d \) above

\( \tilde{e} \) similarly for \( \tilde{e} \), own section of \( \tilde{\varepsilon} \) - limit of \( e \).

This gives us exact triangle: indeed
\[
\begin{align*}
\tilde{d} &= \mu^2(c, \cdot) \\
\tilde{e} &= \mu^2(k(\cdot), \cdot) + \mu^3(c, \cdot, \cdot)
\end{align*}
\]
give us \( \forall Q \),
\[
\begin{align*}
\text{CF}(L_0, L_1) @ CF(Q, L_0) @ CF(Q, L_1) \xrightarrow{\mu^2} CF(Q, L_1) \xrightarrow{\tilde{e}} CF(Q, L_2)
\end{align*}
\]
& this complex behaves nicely with \( Q \).
In fact, these complexes assemble into a twisted complex of Yoneda modules

\[ \mathcal{L}_i \rightarrow \mathcal{L}_i = \text{CF}(\mathcal{L}_i, \mathcal{L}_j) \quad i = 0, 1, 2. \]

\[ \text{CF}(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathcal{L}_0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_2 \]

\[ \text{twisted} \]

\[ \mathcal{L}_2 \]

So now let \( K = \text{Core} \left( \text{CF}(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathcal{L}_0 \rightarrow \mathcal{L}_1 \right) \)

\[ (c, k) \]

\[ \mathcal{L}_2 \]

\[ (c, k) \]

\[ \text{is a closed morphism of twisted complexes} \]

\[ \text{it is a quasi-iso. (since complex } \mathcal{K} \rightarrow \mathcal{L}_2 \text{ quasi-iso) (hence invertible).} \]

By def. we have exact triangles

\[ \text{CF}(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathcal{L}_0 \rightarrow \mathcal{L}_1 \]

\[ \mathcal{K} \]

and similarly via \((c, k)\),

\[ \text{CF}(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathcal{L}_0 \rightarrow \mathcal{L}_1 \]

\[ \mathcal{L}_2 \]

\[ c \]