Recall, a **Lefschetz fibration** is a map 
\[ \pi : (E, \omega, J) \to (S, j) \]
where 
- \( (S, j) \) is a Riemann surface with corners
- \( (E, \omega, J) \) is holomorphic
- \( \pi \) is a holomorphic map away from isolated critical points
- The critical values are distinct
- The local model at critical points is 
  \[ Q : \mathbb{C}^{n+1} \to \mathbb{C} \]
  \[ (z_1, \ldots, z_{n+1}) \mapsto \sum z_i^2 \]
- The fibres are transverse to \( \partial S \)

**Parallel transport:** \( T^h = (\text{Ker} \, \text{d}\pi)^\perp \) horizontal bundle (outside critical points)

- Parallel transport induces an exact symplectic 2-form between fibres.

**Vanishing paths, vanishing cycles, thimble:**

\[ \gamma : [0,1] \to S , \quad t \mapsto \text{crit. value} \]

\[ \gamma \text{ vanishing cycle}: \quad V_\gamma \subset E_{\gamma(0)} \text{ exterior layer sphere} \]

- Points whose parallel transport is a critical point

**Thimble:** \( \Delta_\gamma \subset E \) layer ball, \( \partial \Delta_\gamma = V_\gamma \)

(= union of parallel transport of \( V_\gamma \) along \( \gamma \))

**Local model:**

\[ \mathbb{C}^{n+1} \]

\[ Q \]

\[ C \]

\[ \gamma : [0,c] \subset \mathbb{R}^+ \]

**Small fibres:**

\[ \{ z_i^2 = c \} \subset T^*S^n \]

\[ V_\gamma \subset \text{zero section in } T^*S^n \]

\[ \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} / \sum x_i^2 = c^2 \} \]

and \( \Delta_\gamma = \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} / \| x \|^2 < c^2 \} \)

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**Lemma:**

- \( \gamma \) is a critical path in \( S = \text{crit.}(\pi) \), \( L \subset E \) layer.
  \[ \pi(L) = \gamma \]

- \( L \cap E_{\gamma(0)} \) is a layer in \( E_{\gamma(1)} \), and \( L \) obtained from it by parallel transport along \( \gamma \).

**Proof:** Let \( L \cap E_{\gamma(0)} \) be a layer, isotropic, with the remaining directions \( V \in (T^\perp L) = T^\perp E_{\gamma(1)} \) for generic \( t \) (regular value of \( \pi_L \)).
Matching cycles:

\[ \gamma = \gamma' \cup \gamma'' \text{ embedded}, \quad \gamma', \gamma'' \text{ vanishing paths} \]

\[ \gamma'(0) = \gamma''(0) = q \]

\[ \Rightarrow V_{\gamma'}, V_{\gamma''} \subset E \text{ lift to } \gamma \]

1. If \( V_{\gamma'} = V_{\gamma''} \) then \( \Sigma_\gamma = \Delta_{\gamma'} \cup \Delta_{\gamma''} \) smooth lift \( S^{n-1} \subset E \)

2. Non generally, if \( V_{\gamma'} \neq V_{\gamma''} \) in \( E \), then can modify \( \omega \) by Hamiltonian

an exact deformation to make them match \( \Rightarrow \Sigma_\gamma \subset E \)

Or, by Roser, deforming \( \pi \) by deforming \( \pi \) to get matching.

Call \( \Sigma_\gamma \) a matching cycle

**Example**

\[ E = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = c \right\} \]

\( \cong T^*S^n \)

\( \rightarrow \mathbb{C}^n \) (or function)

Lefschetz fibration: fiber \( p^{-1}(y) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} z_i^2 = c - y^2 \right\} \)

2 sing fibers: at \( y = \pm \sqrt{c} \).

The segment \([-\sqrt{c}, \sqrt{c}]\) is a matching path,
matching cycle \( = \) the sphere \( \sqrt{c} \cdot S^n \subset E \).

What happens as \( y = c \) ? for \( c \to 0 \), \( \quad \Rightarrow \quad \Rightarrow \quad \).

See higher dim local model from Haw's drafts way!
we've viewed each fiber of \( \mathcal{Q} : \mathbb{C}^{n+1} \to \mathbb{C} \) as itself a

Lefschetz fibration w/ \( p : \mathcal{Q}^{-1}(c) \to \mathbb{C} \):

Special cells has a "bi-fibration" & it's a natural source of matching paths

#### Dehn twists:

\( V \subset (M, \omega = d\theta) \) exact Lag. sphere

(parametrized, ie: \( \alpha \cdot \mathbb{S}^n = V \))

\[ \tau: V \subset \text{Symp}(M) \text{ exact symplect. supp \ in nbd}(V) \]

(canonically up to Ham. isom)^

In dim 1:

Observe: this is rescaled geodesic flow \( \mathbb{S}^1 \) w/ standard metric

\( T\mathbb{S}^1 \cong T^*\mathbb{S}^1 \), geodesic flow:

\[ v \mapsto g(v) \quad \text{in dir. of } v. \]

It's Hamiltonian! ... kind of

\[ H(x, v) = h(|v|) \text{ gives: parallel transport by distance } h'(|v|). \]

eg. \( |v|^2 \) gives \( \parallel \) transport by distance \( |v| \).

So: take \( H = h(|v|) \),

\[ \text{This gives } \parallel \text{ transport by amount } 7 \text{ to } 3 \pi \text{ an sympl. 0 section \& induce antipodal map on zero section.} \]

(better) take \( H = \begin{cases} \pi & \text{if } v \neq 0, \\ 0 & \text{if } v = 0 \end{cases} \)

& compose with antipodal \( (x, v) \mapsto (-x, v) \).
Remark: in dim 2, e.g. on $T^*S^2$, $\tau^2_V$ is isotopic to $\text{Id}$ in $\text{Diff}(M)$ [classical]

but in general not in $\text{Symp}(M)$ [Seidel: $HF(L_0, \tau^2_V(L_1))$ \[\neq HF(L_0, L_1)$].

Prop: Non-degeneracy of $L$-Aperiodism around a critical value is

Hamiltonian isotopic to $TV$ when $V = \text{vanishing cycle}$

$\exists \gamma, L \subset T_M \text{ harmonic Whitney sphere}
\text{ inducing } h_\gamma \in \text{Sym}(\Omega,\Omega')$ \[\gamma = \pi^{-1}(q)\].

$h_\gamma \sim TV$

(can see it by explicit computation on local model, or from bifurcation picture:

\[\infty \to Q^{-1}(c) \xrightarrow{\text{proj. to }} \bigtimes_{n+1} \frac{x}{x \pm Vc} \bigtimes \xrightarrow{\text{critical points}} \bigtimes \]

As $c$ goes around origin, $\bigtimes$)

\[\implies \]

Def: a distinguished basis of vanishing paths $\gamma = (\gamma_1, \ldots, \gamma_m)$

\[\bigtimes_{n+1} \frac{x}{x \pm Vc} \bigtimes \]

at or near boundary

$S = D^2$

To the associate ordered sequence of $V_k$'s $(V_1, \ldots, V_m) \subset M = \pi^{-1}(\ast)$

Notation: when $S = D^2$, any two bases of vanishing paths are related

(upto isotopy) by "Hurwitz move"

\[\delta_k \xrightarrow{\text{Hurwitz move}} \delta_{k+1} \overset{\ldots}{\longrightarrow} \delta_k \xrightarrow{\text{Hurwitz move}} \delta_{k+1} \overset{\ldots}{\longrightarrow} \delta_k \]

$& \text{inverse move}$

induce \[ (V_1, \ldots, V_k, V_{k+1}, \ldots, V_m) \overset{\tau_k^{-1}}{\longrightarrow} (V_1, \ldots, \tau V_k(V_{k+1}), V_k, \ldots, V_m) \]

(converse: $(\ldots, V_k, V_{k+1}, \ldots) \overset{\tau_k^{-1}}{\longrightarrow} (\ldots, V_{k+1}, \tau^{-1} V_{k+1}(V_k), \ldots)$)
The Hurwitz equiv. class of \((V_1,...,V_m)\) is an inv. of the Lefschetz\n
fiberation over \(D^2\). Conversely, given any collection of exact Lcp. spheres\n\((V_1,...,V_m)\) in exact sympl. mfld \(M\), can build a L. fibration over \(D^2\)\n
with vanishing cycles \(V_1,...,V_m\), unique up to exact symplectic deformation.\n
contraction: start with \(M \times D^2\), glue local models near copy of \(V_i\) in fibers\n
at boundary, and enlarge/round corners.\n
\((\Leftarrow \text{top: attach a standard "Heinzel" handle along legendrian sphere } V_i \times \{\text{pt}\} \in \mathcal{D}(M \times D^2)\)\n
* To build "exotic" sympl. mflds: start w/ \(M\) containing many interesting lagragn\n
spheres, incl. some that are smoothly ishtopic but not ham. iso., and\n
custom-build L. fibrations w/ fibers \(M\).

Typically, this is done using \(M\) itself carrying a L. fibration:

\[ M \xrightarrow{p} D^2, \text{ lagr. spheres } = \text{ matching spheres for } p. \]

**Ex.** if all paths match,

\[ \Sigma_{\delta^+} = \tau_{\Sigma_{\delta^1}}(\Sigma_{\delta^2}) \]

\[ \Sigma_{\delta^-} = \tau_{\Sigma_{\delta^1}}^{-1}(\Sigma_{\delta^2}) \]

so eg. if \(d_{\text{max}} M = 4\), we get \(\Sigma_{\delta^+} \sim \Sigma_{\delta^-}\)

but not nec. lagr. iso!