Lebesgue Fibration: \( \pi: (E, \omega) \to B \) 

- synplectic Riem. surface (for \( m = 2 \) or \( C \))

- st. \( \{ \) submersion outside of isolated crit. pts, with std. local model \( \) 
  - fibers are sympl. submfd.

1. Lebesgue fibrations:

\[ \text{Consider:} \quad \text{exact (Liouville) symplectic mfd. with corners} \]

- so fibers & \( B \) can have bndry.
  - Equip w/ acc. st. \( (E, J) \overset{\pi}{\to} (B, \delta) \) is \( (J, \delta) \)-holomorphic

\( (J \) is not generic! but \( J \) fibers can be). \( \text{Need convexity: } J \text{-hol. conv. can't escape (max principle).} \)

= automatically, fibers of \( \pi \) are sympl. submfd.

\[ \text{Symplectic connection:} \quad \text{for } x \in \text{crit. of } \pi, \quad T_{E_x} = T_{E_x}^\nu \oplus T_{E_x}^1 \]

\( k \in dt \quad (T_{E_x}^\nu) \quad ^\nu \)

\( (= \text{by } h \text{ fiber}) \)

- Note: \( \delta E = \delta V E \cup \delta h E \quad \delta h E = \text{boundary of fibers} \)

\[ \delta V E = \pi^{-1}(\partial B) \]

2. Lebesgue fibration: \( \pi: (E, \nu, J) \to (S, u_0, \delta) \) Riemann surface w/ bndry

- \( (J, \delta) \)-holomorphic \( \Rightarrow \) fibers symplectic

- \( \pi \) is a submersion outside of a finite set of isolated, independent critical pts

- While \( 3 \) local holom. coord. with \( \pi: (\mathbb{E}_1, \ldots, \mathbb{E}_n) \to \Sigma \mathbb{E} \)

- We'll always assume \( E, S \) are exact symplectic

- Technical conditions: critical pts are in the int. of \( E \)

- \( \delta h E \) is horizontal \( \Rightarrow \) if \( x \in \delta h E, \quad T_{E_x}^1 \subset T_x(\delta h E) \)

- Symp. points can do parallel transport w/ \( T_{E_x}^1 \) safely!

- for simplicity, assume critical values are distinct.
Parallel transport:

\[ E_{g(0)} = \pi^{-1}(g(0)) \]

\[ x^* : [0, 1) \times S \rightarrow \text{ctv}(\pi) \]

\[ \Rightarrow P_\theta : E_{g(\alpha)} \sim E_{g(\varphi)} \]

\[ \text{isomorphism (exact)} \]

\[ \text{because: } \]

\[ \text{horizontal sides } \]

\[ \Rightarrow \int_0^1 \omega = 0 \]

\[ \text{and for } \theta \]

\[ \int_0^1 \omega = \int_0^1 \omega. \]

\[ \text{Exact: } Q : C^{n+1} \rightarrow C \]

\[ (z_1, \ldots, z_{n+1}) \rightarrow \sum z_i^2 \]

need to truncate to get something w/ boundary.

Namely

\[ E = \{ z \in C^{n+1} \mid |Q(z)| \leq r, \quad k(z) \leq s \} \]

\[ \downarrow_{Q} \]

\[ D^2(r) \]

\[ k(z) = \frac{1}{4} \left( |z|^4 - \left| \sum z_i^2 \right|^2 \right) \]

Check:

\[ T^*E^\perp_z = C \cdot (\overline{z}_1, \ldots, \overline{z}_{n+1}) \]

\[ \frac{1}{2} \left| \sum z_i v_i + \overline{z}_i \overline{v}_i \right|^2 \]

\[ \text{and indeed } \frac{dk(z)}{dz} = 0 \]

\[ \frac{dk(i \overline{z})}{dz} = 0 \]

so this is horizontal.

**Fact:** Level sets of \( Q \) are \( \subset T^*S^n \) or after truncation,

\[ DT^*S^n = \{ (v, u) \mid |u| \leq s \} \] Namely,

\[ T^*S^n = \{ (v, x) \in \mathbb{R}^{n+1} \times S^n \mid \langle v, x \rangle = 0 \} \]

\[ c = dv \times dx \]

\[ Q^{-1}(c) \]

\[ c \in \mathbb{R}^+ \]

\[ \left( -|Re(z)|, \frac{Re(z)}{|Re(z)|} \right) \]

\[ (\text{obtain: } Re Q(z) = |Re z|^2 - \text{Im } z^2) \]

\[ In Q(z) = \langle Re z, Im z \rangle \]

(\& these identifications convert \( c \) parallel transport along \( R^+ \))

since

\[ d(\langle -|Re(z)|, \text{Im } z \rangle)(v) = -\frac{\langle v, Re z \rangle}{|Re z|} \text{Im } z - |Re z| \text{Im } v \]

apply to \( v = \overline{z} \).

\[ Q^{-1}(0) \text{ singular at origin} \]

namely identification fails along zero section, \( \text{i.e. } S^0 \subset T^*S^n \) collapsed to critical point.
V_0 \subset E_0(0) = \text{set of pts st. parallel transport along } \gamma \\
\text{converges to cut pt. } p \in E_0(1) \\
V_0 \text{ is an (exact) Lagrangian sphere in } E_0(0) \\
\text{(lap. since collars under } || \text{ transport } = 0 \text{) } \omega|_{V_0} = c/\mu = 0 \\
\text{sphere: by local model} \\
\text{smoothness at } p \text{ not obvious! follows from local model} \\
 \Delta_0 \subset E_0 = \text{union of parallel transport images of } \\
V_0 \text{ along } \gamma \\
\text{small Lag. } B^{n+1} \text{ in } E_0, \ \Delta_0 = V_0. \\
\text{(smoothness at } p \text{ not obvious! follows from local model) \\
Local \text{ model, } \mathbb{C}^{n+1} \qquad \Rightarrow V_0 = \text{zero section in } T^*S^n \\
\mathbb{C} \xrightarrow{\mathcal{O}} \mathbb{C}^{n+1} \\
\gamma = (0, c) \subset \mathbb{C}^n \\
= \{(z) \in \mathbb{C}^n | z = 0, \ Q(z) = c\} \\
\mathbb{R}^{n+1} \text{ with } x_i = c \text{ } \forall i \\
= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_i^2 = c^2 \} \\
\text{and } \Delta_0 = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | |x|^2 \leq c \}. \\
\text{Then if } \gamma(0) = c \notin \mathbb{R}^+ : V_0 = \sqrt{c} \cdot S^n, \ \text{and } \Delta_0 = \bigcup \mathbb{V}_0^{\gamma(t)} S^n.