Let $\pi : E \to D^2$ be a Lefschetz fibration exact, with usual assumptions.

The thimble that lies above $x_j$ we call $A_j$.

$M = \text{fiber above reference point}$

$E$ vanishing cycle in the picture we have fixed a reference fiber $M$ in $E$, we fixed a basis of vanishing paths $\gamma_{ij}$ on which join the basepoint to a critical value.

For each critical value we have a path $k$ along that path we have a thimble $A_j$. [This is a useful way to build $\text{Lag submanifolds}$ in the fiber & the total space.]

Today: Define $\mathcal{F}(\pi)$ and see how it relates to $\mathcal{F}(E)$ & $\mathcal{F}(M)$.

Fukaya Category of a Lefschetz Fibration a.k.a. "Fukaya-Seidel Category"

The objects in this category should consist of Lagrangians in the total space $k$ we want to include also a few select noncompact Lagrangians including thimbles.

Idea: $\mathcal{F}(\pi)$: objects are compact, closed exact Lagrangians $\text{submanifolds of } E$ (i.e. the objects of $\mathcal{F}(E)$)

- graded spin submanifolds with whole package needed to do Floer theory.

- allow thimbles of $\pi$ ($\Delta_1, \ldots, \Delta_m$ or any other thimbles for any vanishing path).
How do we define Floer theory for the whole thing?

For intersecting closed exact Lagrangians, this we've discussed in previous lectures, here in $\mathcal{F}(\mathcal{E})$.

If dealing with closed exact Lagrangian & thimble of $F$, we're also in good shape:

All the intersection points will live in the interior. By Max principle argument, any holomorphic disc will also stay in the interior so we don't need to be concerned about noncompactness of the thimble $\Delta_Y$.

In summary:

- $\text{CF}(L, L')$ OK ($H(\mathcal{E})$)
- $\text{CF}(L, \Delta)$ OK, by Max principle
- Intersection $\rightarrow$ interior (disc) & $\partial_D$ do discs.
- If we have two thimbles and I want to define their Floer theory, what is $\text{CF}(\Delta_0, \Delta_1)$?

The difficulty is, since thimbles are Lagrangian submanifolds with $b_1$ high, they tend to intersect mostly on the $b_1$ fiber. But we can push things a little to get the intersection off the $b_1$ fiber (since Floer theory should be invariant under isotopies & small deformations). However, when we push things a little, 2 things can happen:

- Push intersection into interior
- or push it outside past the boundary.

So we need a rule to how to do this.

Naive Rule: Push endpoint of $\xi_0$ in the $+$ direction.
CASE I:

\[ \Delta_0 \rightarrow \Delta_i \rightarrow \Delta \rightarrow M \rightarrow \Delta_i \rightarrow \Delta_0 \]

Pushing up \( \Delta_0 \) counterclockwise to endpoint further along boundary.

Cross-section of fiber that projects to intersection \( Y_0 \& Y_1 \).

Intersection points are in the interior of the fiber.

CASE II:

\[ Y_0 \rightarrow Y_1 \rightarrow Y_0 \]

Pushing up \( \Delta_0 \) counterclockwise.

In this case we declare that there are no intersections.

Rule (derived): Count intersection \( Y_0 \cap Y_1 \) in the reference fiber \( M \) only if \( \Delta_i \) starts clockwise from \( Y_0 \).

[Now apparent why important that reference point at boundary, we know what it means to be clockwise from each other there.]

Directed category of a family of thimbles:

Given \((\Delta_0, \ldots, \Delta_m)\) we have thimbles \((\Delta_1, \ldots, \Delta_m)\).

We define (arbitrarily) \( \rightarrow (X_0, X) \) has objects \( \Delta_1, \ldots, \Delta_m \).

\( \rightarrow \) indicates 'directed' i.e. directed ordering of objects and morphisms only go forward in the sequence.
Define \( h_m(\Delta_i, \Delta_j) = \begin{cases} 
0 & \text{if } i > j \\
1K \cdot e_i & \text{if } i = j \\
\text{CF}(\nu_i, \nu_j) & \text{if } i < j 
\end{cases} \)

\(i \geq j\)  \hspace{2cm} \text{CASE II situation. \(\nu_i, \nu_j\)} \hspace{2cm} \nu_j \text{ is parked up so don't interact.} \\
\text{upstairs:} \\
insection point \\
1 \text{ thumble, push it up} \\
what remains is the one intersection point \\
The 2 paths at the 2 thumbles intersect transversally at the c.i.p. \\
e_i \text{ - for identity of this object.} \\
1K \text{ - the field.} \\
\(i < j\)  \hspace{2cm} \text{CASE I situation.} \hspace{2cm} \text{Pick up intersection blown vanishing cycle inside that fiber.} \\

\text{Note: \(\text{CF}(\nu_i, \nu_j)\) is combinatorial, involves data in the fibers.} 

\(m^k \text{ in } F[\{e_i\}]: \hspace{2cm} \text{Hom}(\Delta_{i_{k-1}}, \Delta_{i_k}) \otimes \ldots \otimes \text{Hom}(\Delta_{i_0}, \Delta_{i_1}) \rightarrow \text{Hom}(\Delta_{i_0}, \Delta_{i_k}) \] \\
\text{\(m^k\) non-zero unless } i_0 \leq i_1 \leq \ldots \leq i_k \text{ (very clear, otherwise there are no morphisms for \(\nu_j\).)} \\
\text{\(e_i\) strict unit: } m^2(x, e_i) = m^2(e_i, x) = x \text{ (compare)} \\
m^k(x, \ldots, e_i, \ldots) = 0 \\
\text{General case: } i_0 \leq \ldots \leq i_k, m^k \text{ in } F(H) \)
The philosophy for defining \( \mathcal{F}(\mathcal{E}) \):

Take \( \mathcal{F}(\mathcal{E}) \) & take this derived category \( \mathcal{F}(\mathcal{E};\mathcal{M}) \) put them together into a single thing.

"Make the Lagrangians & the blades play with each other." — DA.

The actual depth of \( \mathcal{F}(\mathcal{E}) \) involves a double cover (trick).

Set \( \tilde{\mathcal{E}} = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid y^2 = \pi(x) - 1\} \)

\( \tilde{\mathcal{E}} \) is clearly a double cover of \( \mathcal{E} \). (For each value of \( y \) there are 2 values of \( x \).) Branched at \( \pi^{-1}(1) \cong \mathbb{Z} \).

\( \tilde{\pi} = \text{projection using } y \) is a Lefschetz fibration.

Later can truncate to get \( \mathbb{D}^2 \).

Why is it a Lefschetz fibration?

\[ \tilde{\pi}^{-1}(y) = (x, y) = \left( \pi^{-1}(y^2 + 1), y \right) \]

s.t. \( y^2 = \pi(x) - 1 \)

The fiber in \( \tilde{\mathcal{E}} \) above \( y \) is \( \tilde{\pi}^{-1}(y^2 + 1) \).

All vanishing is taking the double cover of the disc.

\[ y \rightarrow y^2 + 1 \]

This is a fib. with same fiber as old one but twice as many singular fibers & they just got duplicated about the reference pt.

\[ \tilde{\mathcal{E}} \rightarrow \mathbb{C} \text{ is a Lefschetz fibration} \]

with \( \tilde{\pi}^{-1}(0) = \mathbb{Z} \), critical values = \( \{y^2 + 1 \text{ critical at } (0, y)\} \)

cell carries a \( \mathbb{Z}_2 \) action

(\( y \rightarrow -y \))

Switch the 2 halves of
2 kinds of $\frac{3}{2}$-equivariant Lagrangians

- type (U): $L \subset \text{int}(E) \implies \tilde{L} = \text{lift of } L \text{ to } \tilde{E} \implies \tilde{L} + \nu \tilde{L}$ (disjoint, 2 copies of $L$)

- type (B): $\tilde{\Delta} = \text{lift of thimble } \Delta \text{ to } \tilde{E}$ (smooth Lag.-sphere in $\tilde{E}$)

Consisting of 2 copies of the thimble left & right.

Denote $\mathcal{F}(\Pi) =$ subcategory of $\mathcal{F}(\tilde{E})$ with:

- objects: type (U) & type (B) double lifts of Lagrangians in $\tilde{E}$
- morphisms: $\frac{3}{2}$-invariant part of thimbles.

$\text{Hom} \mathcal{F}(\tilde{E}) = \mathcal{C} \tilde{E}$ needs $\text{char}(1k) \neq 2$. 

\[\begin{align*}
\]
 Miracle: $\mathbb{Z}_2$-action on $CP(\overline{\Delta}_1, \overline{\Delta}_2)$ is identity. 

Now that we've defined this category, how does it relate to other things?

$F^*(\overline{\mathbb{V}}_{\overline{\mathbb{V}}}) \times F(E)$ are 2 full & faithful subcategories of $F(\pi)$.

$F^*(\overline{\mathbb{V}}_{\overline{\mathbb{V}}}) \to F(\pi)$

$\Delta_i \to \overline{\Delta}_i$

$F(E) \to F(\pi)$

Punchline: These objects generate the whole thing.

Observe $E$ is itself a fiber of a sheaf of a sheaf of a sheaf.

$E = \{ (x, y, w) \in E \times \mathbb{C} \mid y^2 = \pi(x) - w \} \subseteq E \times \mathbb{C}$

$w = \pi(x) - y^2$

critical points $= (crit \pi) \times \mathbb{C}$

critical values $= crit values (\pi)$

Fiber looks like $\hat{E}_w = \{ (x, y) \in E \times \mathbb{C} \mid y^2 = \pi(x) - w \}$
What does it look like?

Old Lefschetz Fib.

As $w$ approaches a critical value, this double path gets photon.

Send $w$ to origin & then do a $\sqrt{-1}$ call it $y$.

Claim: Fiber is $\sim \hat{E}$

Critical values = critical values ($\pi$)

- $\tau_{\Delta_c}$, vanishing path $\tau_{\Delta_c}$, vanishing cycle $\Delta_c \subset \hat{E}$

- $\tau_{\Delta_1} \cdots \tau_{\Delta_m}$ total monodromy switches the $2$ values

Lemma:

$\tau_{\Delta_1} \cdots \tau_{\Delta_m} (\hat{E}_+) \cong \hat{E}_- [1]$ in $\mathcal{F}(\hat{E})$.

Recall: $\tau_{\Delta_1} \cdots \tau_{\Delta_m} (\hat{E}_+) \cong$ twisted complex
\( CF(\tilde{\Delta}_m, \tilde{L}_+) \otimes \tilde{\Delta}_m \to \tilde{L}_+ \)
\[
\text{(mapping cone of twisted complex)}
\]

\[
\text{exact triangle in } Tw F(E)
\]

Since \( \text{hom}(\tilde{L}_+, \tilde{L}_-[1]) = 0 \) (\( \tilde{L}_+ \cap \tilde{L}_- = \phi \))

get \( \square \subset \tilde{L}_+ \oplus \tilde{L}_- \sim \tilde{L} \)

\( \text{quasi-isom. in } Tw F(E) \)

Pass to \( \mathbb{Z}_2 \)-invariant part \( \Rightarrow \)

in \( Tw F(\mathbb{H}) \), \( \square \subset \tilde{L} \).

\( \text{I.e. } \tilde{L} \text{ is generated by } \tilde{\Delta}_1 \ldots \tilde{\Delta}_m \).

\( \text{Con. } \Rightarrow F(\mathbb{H}) \text{ is generated by } \tilde{\Delta}_1 \ldots \tilde{\Delta}_m \).

\( \text{[We showed how the compact Lagrangians are expressed in terms of } \tilde{\Delta}_1 \ldots \tilde{\Delta}_m] \)

\( \text{Con. } 2: F(E) \to Tw F(\mathbb{H}) \sim Tw F^{-}(E \otimes \mathbb{Z}_2) \)