A category (not necessarily) on a c.r.f.: field $K :=$

- set of objects $Ob A$
- graded vector space $\text{hom}_A(x, y)$ $\forall x, y \in Ob A$
- $Vd \geq 1$, $\mu^d: \text{hom}_A(x_{d-1}, x_d) \otimes \cdots \otimes \text{hom}_A(x_0, x_1) \to \text{hom}_A(x_0, x_d)$ $[2-d]$ shift grading by $2d$:

$$\text{deg} \mu^d(a_1 \ldots a_3) = \Sigma \text{deg} a_i + 2d.$$

$$\text{str}. V a, \ldots a_d, \sum (-1)^{l} \mu^d-l+1(a_{d-l}, \ldots a_{d+1}, \mu^l(a_{k+1}, \ldots a_{k+l}), a_k \ldots a_1) = 0 \quad w/ \text{sign. } \tau = \text{deg}(a_1) + \cdots + \text{deg}(a_k) - k.$$

So:

- $\mu^1(\mu^1(a)) = 0$; $\mu^1$ is a different on hom space.
- $\mu^1(\mu^2(b, a)) = \pm \mu^2(\mu^1(b), a) = \mu^2(b, \mu^1(a))$ Leibniz rule.
- $\mu^2(c, \mu^2(b, a)) \pm \mu^2(\mu^2(c, b), a) = \pm \mu^4(\mu^3(c, b), a) \pm \mu^3(\mu^1(c, b), a) = \pm \mu^3(c, \mu^3(b, a))$ $\pm \mu^3(c, b, \mu^1(a))$

$\mu^2$ is associative up to a homotopy given by $\mu^3$.

Rule: A key point: in real life, at chain level things aren’t associative on the nose, only up to homotopy.

- Taking cohomology w.r.t $\mu^1$, get a functor graded category $H(A)$ on which composition is associative.

$$\text{hom}_{H(A)}(x, y) = H^0(\text{hom}_A(x, y), \mu^1).$$

Each sign is funny $\Rightarrow$ set $[x, y] = (-1)^{\text{deg}_A[x] \cdot \text{deg}_A[y]} [\mu^0(x, y)]$.

($\text{H}^0(A)$ = only degree 0 when = abelian category).

- $A_0$-cat. with $\mu^d = 0 \forall d \geq 3 \Leftrightarrow$ dg-cat. (up to changing sign...)
- $\mu^d d \geq 3$ partially swines to $H(A)$ as Macgyver products.
Ⅰ Cohomological reality:

.i. say A strictly united if \( \forall x \in \text{Ob}\, A, \exists e_x \in \text{hom}_0^0(x,x) \text{ st.} \)

- \( \mu^1(e_x) = 0 \)
- \( \forall a \in \text{hom}(x,y), (-1)^{dy} \mu^2(e_y, a) = a = \mu^2(a, e_x) \)
- \( \mu^k(\ldots, e_x, \ldots) = 0 \) whenever \( d > 3 \).

Too strong to be useful in real-world.

.ii. weaker: A has homotopy units, i.e., \( e_x \in \text{hom}_0^0(x,x) \)

subject to \( \mu^1(e_x) = 0 \); \( \mu^2(a, e_x) = a + \mu^1(h^{-1}(a)) + h_x(\mu^1(a)) \)

\( h_x \) homotopy for \( e_x \) right unit

Similarly \( \mu^2(e_y, a) \) homotopy \( h_y \)

... higher homotopies ... quite painful.

.iii. even weaker but more tractable & sufficient: cohomology units i.e.

\( e_x \in \text{hom}_0^0(x,x), \mu^1(e_x) = 0, [e_x] \) is a unit in \( H(A) \).

Ⅲ Aco-functors: \( F: A \to B \) consists of a map \( F: \text{Ob}\, A \to \text{Ob}\, B \)

and \( n\)-linear \( F^d: \text{hom}_A(x_{d-1}, x_d) \otimes \ldots \otimes \text{hom}_A(x_0, x_1) \to \text{hom}_B(F(x_0), F(x_1))[1-d] \)
\[ 
\sum_{r} \sum_{s_1 + s_2 = d} \mu_{r}^{s_1}(F^{r}(a_d, \ldots, a_{d-s_2+1}), \ldots, F^{s_1}(a_{d}, \ldots, a_{d-1})) 
\]
\[ 
= \sum_{k,l} (-1)^{l} F^{-d-l+1} (a_d, \ldots, a_{d-k+1}, \mu_{l}(a_{d-k+1}, \ldots, a_{d-1})) 
\]

Pictorially: \( F^d = \sum \) \( \pm \) \( \ldots \)

* Equivalently: \( F \) is an 

\[ T(\mathcal{F}) : T(A\mathcal{B}) \to T(B\mathcal{L}) \]

(\( \Sigma \): any inputs arbitrarily Map \( F \) to end gap)

\( F \) is 

\[ \text{Aoo-functor} \iff T(\mathcal{F}) \text{ chain map} \]

* Observe:

\[ \mu_{r}^{s_2}(F^{r}(b), F^{l}(a)) + \mu_{r}^{s_2}(F^{l}(b), a) = F^{r}(\mu_{d}^{r}(b, a)) + F^{l}(\mu_{d}^{r}(b, a)) \]

\[ = F^{r} \text{ and similar up to homotopy given by } F^2 \text{ etc} \ldots \]

In particular:

\[ H(\mathcal{F}) : H(A) \to H(B) \]

\[ [a] \mapsto [F^1(X)] \]

is an ordinary functor.

* Say \( F \) strictly strict if \( F^1(e) = e \) \( F \) is strictly strict if \( H(\mathcal{F}) \) is strict.

\[ F^d(\vdots, \vdots) = 0 \text{ for } d \geq 2 \]

* Def: \( F \) is a 

\[ \text{quasi-isomorphism} \iff H(\mathcal{F}) \text{ is an isomorphism} \]

\[ \text{strict} \text{ is a quasi-equivalence} \iff H(\mathcal{F}) \text{ equivalence (of strict cats)} \]

Composition of Aoo-functors:

\[ (g \circ F)^d(a_d \ldots a_1) = \sum_{s_1 + s_2 = d} g^s(F^r(a_d \ldots), \ldots, F^{s_1}(a_1)) \]

is strictly associative!!

* Given a functor \( F : H(A) \to H(B) \), can understand obstruction to \( \exists \text{ Aoo-functor } F \) stk.

\( H(\mathcal{F}) = F \) - lives in a certain Hochschild Cohomology gap.

* Aoo-functors \( A \to B \) are the object of an AooCat. Fun(A, B) with

\[ \text{morphisms } = \text{Aoo-pre-mixed transmorphisms: } T \circ \text{hom}_B(F(X), Y) = \text{not just} \]

\[ \text{collection of maps } \]

\[ T^d \text{ hom}_B(F(X), Y) \forall X \in \text{A} \]

but also \( T^d : \text{hom}_A(X_0, X) \to \text{hom}_B(F(X_0), Y(X)) \)

\[ \vdots \]

\( \text{Aoo-nat-trans. if well-behaved } (\leftrightarrow \text{ closed under } \mu_{r}^{s}(a, b)) \)
* Aoo-functors $F,G : A \to B$ which coincide on objects are homotopic if the natural transformation $D = F - G : (D^0 = 0, D^1 = F_d - G_d, \forall d \in I)\text{ is nullhomotopic } D = \mu^1_{\text{Fin}(A,B)}(T)$.

**Homological perturbation lemma:**

If $A$ is Aoo-cat., assume $\forall x,y \in \text{ob}B$ we have:

- a chain complex $(\text{A}(x,y), S^A_{x,y})$
- chain maps $F \circ G$
- a homotopy $T_{ij} : \text{hom}_B(x,y) \sim \text{hom}_A(x,y)$

between $F \circ G$ and $\text{Id}$, i.e.

$$FG - \text{Id} = \mu^1_{\text{B}} T + T_0 \mu^1_{\text{B}}$$

Then $\exists$ Aoo-cat. $A$ with $\text{ob} A = \text{ob} B$

$$\text{hom}_A(x,y) = \text{A}(x,y), \mu^1_A = S^A$$

& Aoo-functors $F : A \to B, G : B \to A$ which are id on objects

& st. $F^\circ = F, G^\circ = G$

& st. $F \circ G$ is homotopic to $\text{Id}$.

(we can determine everything, e.g. $\mu_A$, by explicit recursion formulas).

**Corollary:** \[ B \text{ Aoo-cat. } \implies \exists \text{ quasi-isomphic Aoo-cat } A \text{ with } \mu^1_A = 0 \]

(\"minimal model\" of $B$)

$$\text{ob} A = \text{ob} B, \text{hom}_A(x,y) \cong H^0(\text{hom}_B(x,y), S^B_{x,y}), \mu^1_A = 0, \mu^2_A = \mu^2_B$$

but $\mu^3_A$ may be nonzero even if $B$ is dg-cat.

**Corollary:** Any quasi-isomorph of Aoo-category has an inverse up to homotopy.

(by passing to minimal models when it becomes an iso...).

Also, **Lemma:** A chainologically united $\implies \exists A$ strictly united Aoo-cat. with same objects & morphisms as $A$ & quasi-isomorph $\phi : A \sim A$ s.t. $\phi^0 = \text{id}$ and $\phi^1 = \text{id} : \text{hom}$
\textbf{A-co-module} (right module) (not necessarily) / \textbf{A-co-cot. A}

- covariant functor \( A \to \text{chain complexes} \); more explicitly:

\( M \in \text{mod-} A \iff \forall X \in \text{Ob} A, \ M(X) \text{ graded vector space} \)

- \( \forall k \geq 1, \ \mu^k_m : M(X_k) \otimes \text{hom}(X_{k+1}, X_k) \otimes \text{hom}(X_1, X_2) \to M(X_1) [2-k] \)

\[ \begin{aligned}
\text{eqs:} & \quad \sum_k (-1)^k \mu^k_m \left( \mu^k_n (m, a_{2-k+1}, a_{k+1} \ldots a_1) \right) \\
& \quad + \sum_{k, l} (-1)^{k+l} \mu^k_n (a_{k-l+1}, a_{l+1} \ldots a_k, a_{k+1} \ldots a_1) = 0
\end{aligned} \]

In particular \( \mu^0_m (\mu^0_n (m)) = 0 \) differential

\( \mu^1_m (\mu^1_n (m), a) = \mu^1_m (m, \mu^1_n (a)) \)

Leibniz rule

\( \mu^1_m (m, \mu^1_n (b, a)) = \mu^1_m (\mu^1_n (m, b), a) = -\mu^1_m (\mu^1_n (m, b), a) + \ldots \)

product \( \mu^1_m \) is assoc. up to hom. hyp. \( \mu^2_m \)

(If \( \mu^3 = 0 \), \( \Leftrightarrow \) dg-module or dg-cot.)

* In terms of bar complex:

\[ M \otimes T(A[1]) = \bigoplus_{d} \bigoplus_{n \geq 0 \in A} M(X_n) \otimes \text{hom}(X_{n+1}, X_n) \otimes \text{hom}(X_1, X_2) [d] \]

\[ S^2_n = \sum (-1)^n \mu^1_m \otimes \text{id} + \sum (-1)^n \text{id} \otimes \mu^1_n \otimes \text{id} \]

eq. say \( S^2_n = 0 \).

* Pre-module homomorphisms:

\[ \text{hom}_{\text{mod-} A} (M, N) \equiv t = \{ t^d \}_{d \geq 1}, \quad t^d : M(X_d) \otimes \text{hom}(X_{d+1}, X_d) \otimes \text{hom}(X_1, X_2) \to N(X_1) [2-k-t-d] \]

(without any condition)

\( \Leftrightarrow \) induce \( M \otimes T(A[1]) \to N \otimes T(A[1]) \)

\( \text{iff:} \ ) \ M_{\text{mod}} (t) \text{ vanishes} \iff t \text{ is a module homomorphism} .\]

\( \iff t \text{ induces chain map } M \otimes T(A[1]) \to N \otimes T(A[1]) \).
\[(\mu_{\text{mod}}(t,d))(m,a_{d+1\ldots a_1}) = \sum (-1)^{k+1} \mu_{\text{mod}}^{k\cdot l}(d^{-l}(m,a_{d+1\ldots a_{k+1}}),a_{k\ldots a_1}) \]
\[+ \sum (-1)^{k+1} t^{-k+1}(\mu_{\text{mod}}^{d^{-k+1}}(m,a_{d+1\ldots a_{k+1}}),a_{k\ldots a_1}) \]
\[+ \sum (-1)^{k+1} t^{-k+1}(m,\ldots,\mu_{\text{mod}}^{-d}(\ldots),\ldots,a_1). \]

- **composition**: composition of maps on bar complexes

  \[\mu_{\text{mod}}^{2}(t_2,t_1) \cdot (m,a_{d+1\ldots a_1}) = \sum (-1)^{k+1} t_2^{-k+1}(\mu_{\text{mod}}^{d^{-k+1}}(m,a_{d+1\ldots a_{k+1}}),a_{k\ldots a_1}) \]

  \[N \overset{t_2}{\rightarrow} N \overset{t_1}{\rightarrow} P\]

  **Observe**: $\mu_{\text{mod}}$ is **strictly associative**! Set $\mu_{\text{mod}} = 0$. (i.e. $A_{\text{mod}} / A$ from a dg-category!)

  **Observe**: $\text{mod-}A$ is **strictly unital** ($\text{id}_M$ has $\text{id}_M^1 = \text{id}$, $\text{id}_M^d = 0 \forall d \geq 2$)

## Yoneda embedding

**Def.** On objects: $Y \in \text{ob } A 
\implies \ Y = [Y] \in \text{mod-}A$ defined by

\[Y(X) = \text{hom}_A(X,Y), \text{ and structure maps = right multiplications in } A : \]

\[\mu_{\text{mod}}^{d}(m,a_{d+1\ldots a_1}) = \mu_{A}^{d}(m,a_{d+1\ldots a_1}). \]

\[\text{hom}_A(X_d,Y) \rightarrow \text{hom}_A(X_{d-1},X_d) \rightarrow \text{hom}_A(X_1,X_2) \]

**on morphisms:** $b \in \text{hom}_A(Y_0,Y_1) \rightarrow f^!(b) \in \text{hom}_{\text{mod}}(Y_0,Y_1)$

\[f^!(b)(d)(m,a_{d+1\ldots a_1}) = \mu_{A}^{d}(b,m,a_{d+1\ldots a_1}) \]

\[\text{hom}_A(X_d,Y) \rightarrow \text{hom}_A(X_{d-1},Y) \rightarrow \text{hom}_A(X_1,Y) \]

\[\text{higher maps similarly: } f^!(b_{k\ldots b_1})(d)(m,a_{d+1\ldots a_1}) = \mu_{A}^{d+k}(b_{k\ldots b_1},m,a_{d+1\ldots a_1}) \]

- This how pretty much every n-defense property one can wish for

- **Then**: $f^!$ is full and faithful on homology \((\text{i.e. } b_{k\ldots b_1} \text{hom}(X,Y) \overset{f^!}{\rightarrow} \text{hom}_{\text{mod}}(X,Y))\)
  
  & co-unital (hence "embedding")

- **Contrary**: any co-unital Aoo-cat is quasi-isomorphism to a strictly unital dg-category.
  (namely, image of $f^!$).

- mod-$A$ usually a lot larger than $A$! can take $\Theta$ of objects, mapping cones, $\ldots$!