① Degenerations and the mirror map

We've seen: e_i basis of $H^2(X,\mathbb{Z})$, $e_i \in$ Kähler cone

$\rightsquigarrow$ coords. on complexified Kähler moduli space:

if $[B+i\omega] = \sum t_i e_i$, let $q_i = \exp(2\pi i t_i) \in \mathbb{C}^*$

$q_i \to 0$ corresponds to <u>large volume limit</u> $(\mathrm{Im}(t_i) \to \infty)$

Physics predicts that the mirror situation = degeneration to "<u>large cx. structure limit</u>" and that, near such a limit point, $\exists$ "canonical coordinates" on cx. moduli space – making it possible to describe the mirror map.

- <u>Degeneration</u>: = family

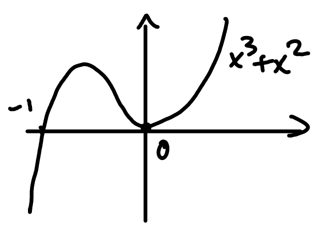
$$\begin{array}{ccc} \overset{\text{complex manifold}}{\mathcal{X}} & \supset & X_t \\ \pi \downarrow & & \downarrow \\ D^2 & \ni & t \end{array}$$

where for $t \neq 0$, $X_t \underset{\text{diff.}}{\cong} X$ (with varying J)

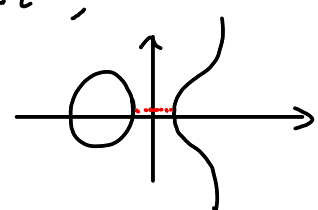
for $t=0$, X_0 typically singular

<u>Ex.</u>: elliptic curves $C_t = \{ y^2 z = x^3 + x^2 z - t z^3 \} \subset \mathbb{CP}^2$

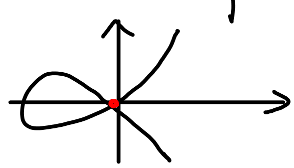
(in affine coords. C_t: $y^2 = x^3 + x^2 - t$,



real part for $t > 0$:

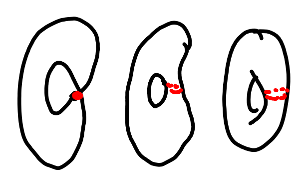


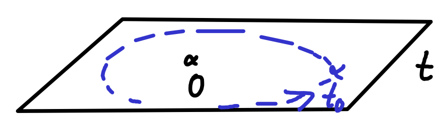
for $t = 0$:



)

C_t smooth for $t \neq 0$, C_0 <u>nodal</u>: actually





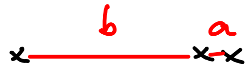
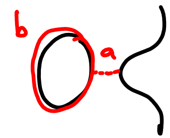
- <u>Monodromy</u>: follow family (X_t) as t varies along loop in $\pi_1(D^2 \setminus \{0\}, t_0)$ going around origin. All X_t's diffeomorphic, so induces a monodromy diffeomorphism φ of X_{t_0}, defined up to isotopy.

Induces: $\varphi_* \in \mathrm{Aut}(H_n(X_{t_0}, \mathbb{Z}))$

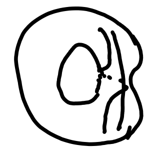
In above example: φ acts on $H_1(C_{t_0}) = \mathbb{Z}^2$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (Dehn twist)

(observe: $C_t \xrightarrow{2:1} \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by projection to x

branch pts are $\begin{cases} \text{roots of } x^3 + x^2 - t \\ \infty \end{cases}$

Near $t \to 0$:  $\Longleftrightarrow$ 

root near -1 2 roots near 0

when t goes around 0, roots near 0 rotate , induces 

$\Rightarrow$ monodromy maps to 

The complex parameter t is ad hoc. A more natural way to describe the degeneration would be to describe C_t as an abstract elliptic curve $C_t \simeq \mathbb{C}/\mathbb{Z} + \tau(t)\mathbb{Z}$

then $\tau(t)$, or rather $\exp(2\pi i \tau)$ it turns out, is a better quantity...

Equip C_t with a holom. vol. form Ω_t normalized so $\int_a \Omega_t = 1 \quad \forall t$

Then let $\tau(t) = \int_b \Omega_t$ ... as t goes around origin, $\underline{\tau(t) \to \tau(t) + 1}$!!

since $b \mapsto b + a$

still, $q(t) = e^{2\pi i \tau(t)}$ is single-valued

As $t \to 0$, $\operatorname{Im} \tau(t) \to \infty$ and $q(t) \to 0$

$\hookrightarrow$ for $t \in \mathbb{R}_+$, $t \to 0$:

$\left.\begin{array}{l} \int_a \frac{dx}{y} \in -i\mathbb{R}^+ \text{ and } \to 0 \\ \int_b \frac{dx}{y} \in \mathbb{R}^+ \text{ and } \to \text{constant} \end{array}\right]$ ratio $\to +i\infty$

$\hookrightarrow$ $q(t)$ holom. function of t, and goes around 0 once when t does ie. has a single root at $t = 0$.

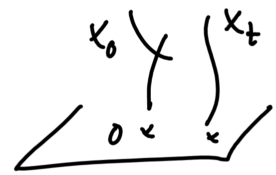
$\Rightarrow$ q is a local coordinate for Family!

Next: analogue of this for a family of Calabi-Yau 3-folds?

③ <u>Degeneration & monodromy:</u> (linear algebra)

$$\begin{array}{ccc} \mathcal{X} & \supset & X_t \\ \downarrow & & \downarrow \\ D^2 & \ni & t \end{array}$$

family of compact Kähler manifolds, X_t smooth, X_0 singular



We've seen: <u>monodromy</u> around $t=0$ induces $\varphi_* \in \mathrm{Aut}(H^n(X_{t_0}, \mathbb{Z}))$

<u>Thm:</u>
- all eigenvalues of φ_* are roots of unity. Thus $\exists\, N, k$ st. $(\varphi_*^N - Id)^k = 0$
- moreover, $k \leq n+1$.

- replacing φ by φ^N ("base change": $X'_t = X_{t^N}$), can assume φ_* is <u>unipotent</u> ie. $(\varphi_* - id)^k = 0$; <u>maximally unipotent</u> $:=$ $k = n+1$.

- Can define a <u>weight filtration</u> associated to unipotent φ_*:
[Comes from Jordan block decomposition of φ_*]

let $N = \log(\varphi_*) = (\varphi_* - id) - \dfrac{(\varphi_* - id)^2}{2} + \dots + (-1)^{n+1}\dfrac{(\varphi_* - id)^n}{n}$

nilpotent $N^{n+1} = 0$ acting on $V = H^n(X, \mathbb{Q})$

$\Rightarrow \exists$ filtration $0 \subseteq W_0 \subseteq W_1 \subseteq \dots \subseteq W_{2n} = V$ s.t.

$$\begin{cases} N(W_i) \subseteq W_{i-2} \\ N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1} \end{cases}$$

<u>Construction:</u>
- $N^n : W_{2n}/W_{2n-1} \xrightarrow{\sim} W_0$ so: $W_0 := \mathrm{Im}(N^n)$, $W_{2n-1} := \ker(N^n)$
- then let $V' = W_{2n-1}/W_0$, N induces $N' \in \mathrm{End}(V')$
(since $W_{2n-1} = \ker N^n \supseteq \mathrm{Im}\, N$ & $W_0 = \mathrm{Im}(N^n) \subseteq \ker N$)

and $(N')^n = 0 \rightsquigarrow$ by induction,

$$\begin{array}{ccccccc} 0 \subseteq & W'_0 & \subseteq & \dots & \subseteq & W'_{2n-2} & = V' \\ & \wr\wr & & & & \wr\wr & \\ & W_1/W_0 & \subseteq & \dots & \subseteq & W_{2n-1}/W_0 & \end{array}$$

and $W_{2n-2} = \{ v \,/\, N^{n-1}(v) \in W_0 = \mathrm{Im}\, N^n \} \supseteq \mathrm{Im}\, N$ so $W_{2n} \xrightarrow{N} W_{2n-2}$

$W_1 = \{ N^{n-1}(v) \,/\, N^n(v) = 0 \} \subseteq \ker N$ so $W_1 \xrightarrow{N} 0$; $W_k \to W_{k-2}$ by induction.