① **Recall:** we're interested in $g=0$, $k=3$ GW invts of Calabi-Yau 3-folds.

For $\deg \alpha_i = 2$, $\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta} = \int_{[\overline{\mathcal{M}}_{0,3}(X,J,\beta)]} ev_1^*\alpha_1 \wedge ev_2^*\alpha_2 \wedge ev_3^*\alpha_3$

- (for $\beta \neq 0$) $= \left(\int_\beta \alpha_1\right)\left(\int_\beta \alpha_2\right)\left(\int_\beta \alpha_3\right) . \#[\overline{\mathcal{M}}_{0,0}(X,J,\beta)]$.

(integrating over part of $\overline{\mathcal{M}}_{0,3}(\dots)$ that corresponds to a fixed rational curve w/ different positions of marked pts)

- if $\beta = 0$, then constant maps only $\Rightarrow \langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,0} = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3$.

$\Rightarrow$ ★ Yukawa Coupling: physicists write

$$\langle \alpha_1,\alpha_2,\alpha_3\rangle = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\substack{\beta\in H_2(X)\\ \beta\neq 0}} \langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta}\, e^{2\pi i\int_\beta \overbrace{B+i\omega}^{\text{complexified Kähler class}}}$$

Better: treat this as a formal power series

$$\langle \alpha_1,\alpha_2,\alpha_3\rangle = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\substack{\beta\in H_2(X)\\ \beta\neq 0}} \langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta}\, q^\beta$$

$\in \Lambda :=$ completion of group ring $\mathbb{Q}[H_2(X)] = \left\{\sum_{\text{finite}} a_{\beta_i} q^{\beta_i}\right\}$
(allowing infinite sums if $\int_{\beta_i}\omega \to +\infty$)

★ Rmk: another way to encode same data is as a product structure on cohomology of X: namely, fix (η_i) basis of $H^*(X)$, and let (η^i) dual basis ie. $\int_X \eta_i\wedge\eta^j = \delta_{ij}$. Then set

$$\alpha_1 * \alpha_2 = \sum_i \langle \alpha_1,\alpha_2,\eta^i\rangle \eta_i = \alpha_1\wedge\alpha_2 + \sum_{\beta\neq 0, i} \langle \alpha_1,\alpha_2,\eta^i\rangle_{0,\beta}\, q^\beta \eta_i$$

Def/Thm: quantum cohomology: $QH^*(X) = (H^*(X;\Lambda), *)$ associative algebra

★ Can view q as a set of coordinates on the complexified Kähler moduli space.

(X,J) complex: Kähler cone $\mathcal{K}(X,J) = \{[\omega] / \omega \text{ Kähler}\} \subset H^{1,1}(X)\cap H^2(X,\mathbb{R})$

This is an open convex cone: (- nondegeneracy is an open condition
- Kähler closed under convex combinations)

$\dim_{\mathbb{R}} \mathcal{K}(X,J) = h^{1,1}(X)$ ... but becomes a $\mathbb{C}$ manifold by adding "B-field"

(2)

Def.: (X,J) CY 3-fold with $h^{1,0}=0$ (so $h^{2,0}=0$, and $H^{1,1}=H^2$)
$\Rightarrow$ the complexified Kähler moduli space:

$$\mathcal{M}_{\text{käh}}(X) := \big(H^2(X,\mathbb{R}) + i\,\mathcal{K}(X,J)\big)/H^2(X,\mathbb{Z})$$
$$= \{[B+i\omega] \,/\, \omega \text{ kähler}\}/H^2(X,\mathbb{Z}).$$

Choose $(e_i)_{i=1..m}$ basis of $H^2(X,\mathbb{Z})$, $e_1 \dots e_m \in \overline{\mathcal{K}(X,J)}$
(exists by openness)

Write $[B+i\omega] = \sum t_i e_i$, $t_i \in \mathbb{C}/\mathbb{Z}$

Then coordinates on $\mathcal{M}_K := \{ q_i = \exp(2\pi i\, t_i) \} \in$ open subset of $(\mathbb{C}^*)^m$ containing $(\mathbb{D}^*)^m$

and $q^\beta \longleftrightarrow q_1^{d_1} \cdots q_m^{d_m}$, where $d_i = \int_\beta e_i$ positive integers
(since e_i kähler, $\beta = [\text{cx. curve}]$)

__Remark__: GW invts vs. enumerative geometry:

Let $N_\beta = \#[\mathcal{M}_{0,0}(X,J,\beta)] \in \mathbb{Q}$, then we've seen that

$$\alpha_1,\alpha_2,\alpha_3 \in H^2(X) \rightsquigarrow \langle \alpha_1,\alpha_2,\alpha_3\rangle = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\beta\neq 0} \langle\alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta}\, q^\beta$$
$$= \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\beta\neq 0} \Big(\int_\beta \alpha_1\Big)\Big(\int_\beta \alpha_2\Big)\Big(\int_\beta \alpha_3\Big)\, N_\beta\, q^\beta$$

Yet the first day I wrote $\displaystyle\sum_{\beta\neq 0} \cdots\cdots\, n_\beta \frac{q^\beta}{1-q^\beta}$ instead

where "n_β = # rational curves in class β"?

Discrepancy is due to expected contributions of __multiple covers__.

- Let $C \subset X$ Calabi-Yau 3-fold be an embedded rational curve ($C \simeq \mathbb{P}^1$).
  By a thm of Grothendieck, any holom. vect. bundle $/\mathbb{P}^1$ splits as direct sum of line bundles: so $NC = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$
  ($\mathcal{O}(d)$ = sections are deg. d homogeneous holom. functions on $\mathbb{C}^2$)
  $\mathcal{O}(-1)$ = tautological line bundle

③ $c_1(TX)\cdot[C] = 0 = c_1(TC)\cdot[C] + c_1(NC)\cdot[C] = 2 + d_1 + d_2$

$\Rightarrow \quad d_1 + d_2 = -2$, "generic case" is $NC = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

Then C is automatically regular as a J-hol. curve.

$\rightsquigarrow$ C contributes 1 to $N_{[C]}$.

Q: contrib^n of mult. covers of C to the GW-invariant $N_{[kC]}$?

$\mathcal{M}(kC) \subset \mathcal{M}_{0,0}(X, J, k[C])$ component consisting of covers of C

(has excess dimension) $\Rightarrow \#[\mathcal{M}(kC)]^{vir}$?

Thm: (Voisin, ...) ‖ If $NC = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ then the contribution of C to $N_{k[C]}$ is $1/k^3$

(NB: $\mathcal{M}(kC) = \{ \mathbb{P}^1 \xrightarrow[P/Q]{k:1} \mathbb{P}^1 \to C \subset X \}/\sim$ is a smooth orbifold of $\dim_{\mathbb{R}} 2k-2$

rational function of deg. k

obstruction sheaf is a rank $2k-2$ vector bundle; Euler class calculation)

NOT SO EASY.

Hence, expect: ‖ $$N_\beta = \sum_{\beta = k\gamma} \frac{1}{k^3} n_\gamma \quad (*)$$

and now, $$\sum_\beta \left(\int_\beta \alpha_1\right)\left(\int_\beta \alpha_2\right)\left(\int_\beta \alpha_3\right) N_\beta\, q^\beta$$

$$= \sum_{\gamma;\, k \geq 1} \left(\int_{k\gamma} \alpha_1\right)\left(\int_{k\gamma} \alpha_2\right)\left(\int_{k\gamma} \alpha_3\right) \frac{n_\gamma}{k^3}\, q^{k\gamma}$$

$$= \sum_\gamma \left(\int_\gamma \alpha_1\right)\left(\int_\gamma \alpha_2\right)\left(\int_\gamma \alpha_3\right) n_\gamma \underbrace{\sum_{k\geq 1} q^{k\gamma}}_{= \frac{q^\gamma}{1-q^\gamma}}$$

However, how to define "n_γ = # curves in class γ", and whether those numbers satisfy $(*)$, is unclear, or at least, outside the scope of this class. See: { Gopakumar-Vafa conj. / Donaldson-Thomas invts, / MNOP conjecture

Instead: take $(*)$ as a definition of n_γ

and hope these might be integers & actual curve counts ....