1 <u>Recall</u>: $\mathcal{M}_{g,k}(X,J,\beta) = \{(\Sigma, j, z_1 \dots z_k),\ u : \Sigma \to X \mid \bar\partial_J u = 0,\ u_*[\Sigma] = \beta\}/\sim$

- <u>transversality</u>: for generic J, <u>simple</u> curves are regular ($\Rightarrow \mathcal{M}^*_{g,k} \subset \mathcal{M}_{g,k}$ smooth)
- <u>compactness</u>: $\mathcal{M}_{g,k}(X,J,\beta)$ can be compactified by adding <u>stable maps</u> (domain = nodal curve)

- Assume that we can achieve transversality, even for non-simple curves.

Then: $\overline{\mathcal{M}}_{g,k}(X,J,\beta) = \left\{ \begin{array}{c} \text{(possibly nodal) } J\text{-hol. curves of genus } g \\ \text{representing class } \beta \end{array} \right\}/\sim$

compact oriented of $\dim_{\mathbb{R}}$ $2d = 2\langle c_1(TX), \beta\rangle + 2(n-3)(1-g) + 2k$

& carries a fundamental class $[\ ] \in H_{2d}(\overline{\mathcal{M}}_{g,k}(X,J,\beta), \mathbb{Q})$ ⚠ $\mathbb{Q}$-coeffs: due to <u>orbifolding</u>

carries evaluation maps $ev_1, \dots, ev_k : \overline{\mathcal{M}}_{g,k}(X,J,\beta) \longrightarrow X$

$$(\Sigma, j, z_1 \dots z_k, u) \longmapsto u(z_i)$$

$\rightsquigarrow$ <u>Gromov-Witten invariants</u> = given $\alpha_1, \dots, \alpha_k \in H^*(X)$, $\Sigma \deg(\alpha_i) = 2d$,

$$\langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,k}(X,J,\beta)]} ev_1^* \alpha_1 \wedge \dots \wedge ev_k^* \alpha_k$$

or equivalently by Poincaré duality, given cycles $C_1, \dots C_k$ $([C_i] = PD(\alpha_i))$ and assuming the evaluation maps are transverse to them, this is equal to $\#(ev_1^{-1}(C_1) \cap \dots \cap ev_k^{-1}(C_k))$ ie. "count of genus g holom. curves in class β passing through $C_1, \dots, C_k$". In general $\in \underline{\mathbb{Q}}$!

- In the case of a Calabi-Yau 3-fold, dim. formula simplifies to $\dim \overline{\mathcal{M}}_{g,k}(X,J,\beta) = 2k$ ... ie, holomorphic curves are <u>isolated</u>.

We'll look at $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta}$ for $\deg(\alpha_i) = 2$.

- More about genus 0 GW invariants of Calabi-Yau 3-folds:

how does one build $[\overline{\mathcal{M}}_{0,k}(X,J,\beta)]$? <u>2 flavors of GW theory</u>:

→ <u>symplectic geometry</u>: • <u>Transversality</u>:

- for simple curves, obtained by choosing J generic (seen last time: $\mathcal{J}_{reg}$ dense in $\mathcal{J}(X,\omega)$)
- multiple covers: always occur with excess dimension $\forall J$ ($\Sigma' \xrightarrow[\pi]{} \Sigma \to X$, deform covering π .... even though curves in CY 3-folds should be isolated!)

Also, multiply covered maps have automorphisms (autom. of covering)

$$\begin{tikzcd}[row sep=small] \Sigma' \arrow[d, "\simeq"'] \arrow[dr, "\pi"] & & \\ \Sigma' \arrow[r, "\pi"'] & \Sigma \arrow[r] & X \end{tikzcd}$$

$\Rightarrow \mathcal{M}_{0,k}(X,J,\beta)$ orbifold

can restore transv. by using domain-dependent J = function

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{J}(X,\omega) \\ \bar{\mathcal{M}}_{0,k} & \end{tikzcd}$$

universal curve

$u:(\Sigma,j) \to X$, $z \in \Sigma$, $du + J(u(z), \underline{z})\, du\, j = 0$

or a perturbation term: $\bar\partial_J u + \nu(z, u(z)) = 0$.

These perturbations "break" the symmetry of the covering π; they're one reason why $\#$curves $\in \mathbb{Q}$ not $\mathbb{Z}$. (see below)

⚠ case of multiply covered bubbles hardest to deal with; fortunately not an issue for us.

• <u>Compactness</u>: we allow stable curves consisting of several components....

→ in general those should contribute codim. 2 to $\bar{\mathcal{M}}$ if regularity holds (ie. $\mathcal{M}_{0,k}(X,J,\beta)$ defines a <u>pseudocycle</u>, good enough)

→ in CY 3-fold case they shouldn't contribute at all.

<u>Point</u>: for generic J, we have transversality for simple curves.

Then, given β, $\exists$ <u>finitely many</u> classes of curves with energy $\le \int_\beta \omega$ (Gromov compactness), and the simple curves in these classes are <u>isolated</u>.

For generic J, evaluation maps are mutually transverse ie. these simple curves are mutually <u>disjoint</u> ... so no bubbled configurations (unless all components cover the same simple curve!)

③ → *algebraic geometry* : prefer to keep J integrable even if it means failure of transversality. [Sometimes transv. *does* hold...]

For integrable J & fixed j, $D_{\bar\partial}$ is complex linear, honest $\bar\partial$ operator on sections of u^*TX – a holom. bundle! Cokernel $= H^1(\Sigma, u^*TX)$

Assume u immersion for simplicity: then $u^*TX = T\Sigma \oplus u^*N$ ← normal bundle

tangent part $H^1(\Sigma, T\Sigma)$ is taken care of by letting j vary (or trivial for $g=0$...). Remaining cokernel: $Obs_u := H^1(\Sigma, u^*N)$.

(if u not immersed, Obs_u = quotient of $H^1(\Sigma, u^*TX)$)

Putting these together over the moduli space of stable maps, get obstruction sheaf $\underline{Obs} \to \overline{\mathcal{M}}_{g,k}(X,J,\beta)$ algebraic space (*stack*)

A perturbation of the holom. curve eqn to $\bar\partial_J u = \nu$ yields a section $\pi_{\mathrm{coker}\,\bar\partial}(\nu)$ of $\underline{Obs}$; the perturbed moduli space is its zero set.

This lets us define a homology class $[\overline{\mathcal{M}}_{g,k}(X,J,\beta)]^{\mathrm{virt}} \in H_{2d}(\overline{\mathcal{M}}_{g,k}(X,J,\beta);\mathbb{Q})$ virt. fundamental class.

E.g. if $\overline{\mathcal{M}}_{g,k}(X,J,\beta)$ smooth but excess dimensional, $\underline{Obs}$ bundle

$$\rightsquigarrow [\overline{\mathcal{M}}_{g,k}(X,J,\beta)]^{\mathrm{virt}} = e(\underline{Obs}) \quad \text{Euler class.}$$

<u>Recall</u>: we're interested in $g=0$, $k=3$ GW invts of Calabi-Yau 3-folds.

For $\deg \alpha_i = 2$, $\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta} = \int_{[\overline{\mathcal{M}}_{0,3}(X,J,\beta)]} ev_1^*\alpha_1 \wedge ev_2^*\alpha_2 \wedge ev_3^*\alpha_3$ ← has $\dim_{\mathbb{R}} = 6$

or, taking dual cycles C_i (of codim. 2) & recalling $\mathcal{M}_{0,3} = \{(S^2,0,1,\infty)\} = pt$:

$$\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta} = \# \overline{\left\{ u: S^2 \to X \text{ J-holom., class } \beta \,\middle/\, \begin{matrix} u(0)\in C_1 \\ u(1) \in C_2 \\ u(\infty)\in C_3 \end{matrix} \right\} /\sim}$$

reparametrization acts transitively on triples of pts. $(C_i\cdot\beta)$ pts of S^2 are mapped to C_i (counting w/ multiplicity)

$$= (C_1\cdot\beta)(C_2\cdot\beta)(C_3\cdot\beta) \; \underbrace{\# \overline{\{u: S^2 \to X \text{ J-hol. class } \beta\}/\sim}}_{N_\beta = \#[\overline{\mathcal{M}}_{0,0}(X,J,\beta)]}$$

(4.)

In other terms: $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} = \left(\int_\beta \alpha_1\right)\left(\int_\beta \alpha_2\right)\left(\int_\beta \alpha_3\right) \cdot \#[\bar{\mathcal{M}}_{0,0}(X,J,\beta)]$.

(perhaps easier to see directly... integrating over part of $\bar{\mathcal{M}}_{0,3}(\dots)$ that corresponds to a fixed rational curve w/ different positions of marked pts)

Except ... if $\beta = 0$, then constant maps only $\Rightarrow$ need $u(0) = u(1) = u(\infty) \in C_1 \cap C_2 \cap C_3$

ie. $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,0} = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3$.

- Yukawa Coupling: physicists write

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\substack{\beta \in H_2(X) \\ \beta \neq 0}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} \, e^{2\pi i \int_\beta \overbrace{B + i\omega}^{\text{complexified Kähler class}}}$$