① **Pseudoholomorphic curves:**

(X^{2n},ω) symplectic manifold, J compatible almost-$\mathbb{C}$ structure
($J^2=-1$, $\omega(\cdot,J\cdot)$ Riem. metric)

(Σ,j) Riemann surface of genus g, $z_1,\dots,z_k\in\Sigma$ marked points

Def: $u:\Sigma\to X$ is <u>J-holomorphic</u> if $J\circ du = du\circ j$
i.e. $\bar\partial_J u = \frac{1}{2}(du + J\,du\,j) = 0 \in \Gamma(\Sigma,\Omega^{0,1}_\Sigma\otimes u^*TX)$

Def. $\mathcal{M}_{g,k}(X,J,\beta) = \left\{(\Sigma,j,z_1,\dots,z_k),\ u:\Sigma\to X \,/\, \begin{matrix} u_*[\Sigma]=\beta \\ \bar\partial_J u = 0\end{matrix}\right\}/\sim$
$\beta\in H_2(X)$

(equivalence relation: $\phi:\Sigma\xrightarrow{\sim}\Sigma'$, $\phi(z_i)=z_i'$, $\begin{tikzcd} \Sigma \arrow[d, "\phi"'] \arrow[r, "u"] & X \\ \Sigma' \arrow[ur, "u'"'] \end{tikzcd}$)

i.e. zero set of a section $\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathrm{Map}(\Sigma,X)_\beta\times\mathcal{M}_{g,k} \arrow[u, bend left, "\bar\partial_J"] \end{tikzcd}$ vector bundle, $\mathcal{E}_u = \Gamma(\Sigma,\Omega^{0,1}_\Sigma\otimes u^*TX)$

More precisely, look at $W^{k+1,p}$ maps, and $\mathcal{E}_u = W^{k,p}(\Sigma,\Omega^{0,1}_\Sigma\otimes u^*TX)$
$\rightsquigarrow$ Banach bundle over a Banach manifold

The linearized operator $D_{\bar\partial}: W^{k+1,p}(\Sigma,u^*TX)\times T\mathcal{M}_{g,k}\to W^{k,p}(\Sigma,\Omega^{0,1}\otimes u^*TX)$
$$D_{\bar\partial}(v,j') = \bar\partial v + \tfrac{1}{2}\nabla_v J\cdot du\circ j + \tfrac{1}{2}J\cdot du\cdot j'$$

$D_{\bar\partial}$ is Fredholm, of $\mathrm{index}_{\mathbb{R}} = 2d := 2\langle c_1(TX),\beta\rangle + n(2-2g) + (6g-6+2k)$

Def: $u\in\mathcal{M}_{g,k}(X,J,\beta)$ is <u>regular</u> if $D_{\bar\partial}$ is <u>onto</u> at u.

Def: $u:\Sigma\to X$ is <u>simple</u> ("somewhere injective") if $\exists\, z\in\Sigma$ st. $\begin{cases} du(z) \text{ injective} \\ u^{-1}(u(z)) = \{z\}\end{cases}$

otherwise, u factors through a covering $\Sigma\to\Sigma'\to X$

$\mathcal{M}^*_{g,k}(X,J,\beta) = \{\text{simple } J\text{-hol. curves}\}$

Thm: $\mathcal{J}^{reg}(X,\beta) = \{J\in\mathcal{J}(X,\omega) \,/\, \text{every simple } J\text{-hol. curve in class } \beta \text{ is regular}\}$
is a Baire subset in $\mathcal{J}(X,\omega)$
For $J\in\mathcal{J}^{reg}(X,\beta)$, $\mathcal{M}^*_{g,k}(X,J,\beta)$ is smooth of real dim. $2d$ and carries a natural orientation.

⚠ in general $\mathcal{M}_{g,k}$ <u>orbifold</u> (Σ with automorphisms)

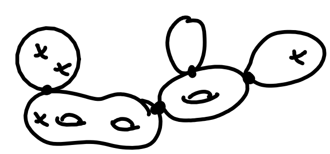
<u>Proof sketch:</u> • consider $\bar\partial_J u = 0$ as eqn on $\mathrm{Map}(\Sigma, X) \times \mathcal{M}_{g,k} \times \mathcal{J}(X,\omega) \ni (u, j, J)$
then linearization is surjective for all simple maps.
(while fails for multiple covers...)
"univ. moduli space" $\widetilde{\mathcal{M}}^* \xrightarrow[\pi_J]{} \mathcal{J}(X,\omega)$: projection to J is Fredholm,
$\Rightarrow$ by Sard-Smale a generic J is a regular value.
and then $\mathcal{M}^*_{g,k}(X,J,\beta)$ is smooth.

• orientation: need orient$^{\underline{n}}$ on $\ker(D_{\bar\partial})$. If J is integrable then $D_{\bar\partial}$ is $\mathbb{C}$-linear and $\exists$ natural orientation. Can still do it in general. ⚠

★ <u>Moreover</u>: $\forall J_0, J_1 \in \mathcal{J}^{reg}(X,\beta)$, $\exists$ (dense set of choices of) path $\{J_t\}_{t\in[0,1]}$ s.t.
$\coprod_{t\in[0,1]} \mathcal{M}^*_{g,k}(X, J_t, \beta)$ smooth cobordism between $\mathcal{M}^*_{g,k}(X,J_0,\beta)$ and $\mathcal{M}^*_{g,k}(X,J_1,\beta)$
but we need a <u>compactness</u> result, else "# curves" not indep^{t} of $J \in \mathcal{J}_{reg}$!

★ So far we haven't really used the symplectic form ω... it's used for

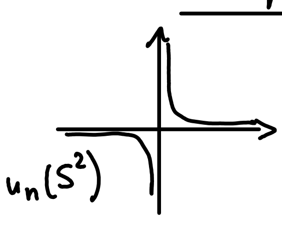
<u>Thm</u>: (<u>Gromov compactness</u>)
$u_n : \Sigma_n \to X$ sequence of J-holom. curves, $J \in \mathcal{J}(X,\omega)$,
$E(u_n) = \int_{\Sigma_n} u_n^*\omega = \langle [\omega], u_{n*}[\Sigma_n] \rangle$ bounded $\Rightarrow$
$\exists$ subsequence that converges to a <u>stable map</u> $u_\infty : \Sigma_\infty \to X$



ie: $\Sigma_\infty = \bigcup$ nodal Riemann surfaces
all marked points & nodes are distinct in the domain
(if they come together, create a constant bubble to keep them separated)

<u>Phenomenon</u>: besides possible degeneration of domain (Σ_n, j_n) to a nodal curve, the main phenomenon is <u>bubbling</u> of spheres

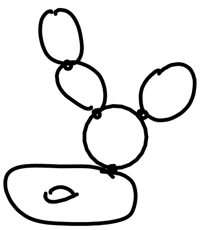
<u>Example</u>: $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$
$(x_0 : x_1) \longmapsto (x_0 : x_1), (n x_1 : x_0)$



(in affine chart $x = x_1/x_0$: $x \longmapsto \left(x, \frac{1}{nx}\right)$ + extend at 0 & ∞)

then away from origin, uniform convergence to $x \longmapsto (x,0)$
so limit seems to be just 1st coord. axis .. missing part!
but if we reparametrize : $\tilde{x} = nx$, then get $\tilde{x} \longmapsto \left(\frac{\tilde{x}}{n}, \frac{1}{\tilde{x}}\right)$
uniform cv away from ∞ to $\tilde{x} \mapsto \left(0, \frac{1}{\tilde{x}}\right) \Rightarrow$ 2nd coord axis ✓

<u>Ideas:</u>
- identify bubbling regions = where $\sup |du_n| \to \infty$
- in those regions, rescale domain : $v_n(z) := u_n(z_n^0 + \varepsilon_n z)$, $\varepsilon_n \to 0$ suitably chosen $\Rightarrow$ a subsequence of v_n converges to a map $v_\infty : \mathbb{C} \to X$, which by removable sing. theorem extends to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$: the bubble !



- intermediate bubbling stages $\Rightarrow$ might need various rescalings to catch all bubbles.
- The process is <u>finite</u> because of energy estimates :

$$E = \int u^* \omega \geq \hbar > 0 \quad \text{for all nonconstant closed J-hol. curves}$$
$\uparrow$ minimum energy

and we've assumed an upper bound on total energy.