① Deformations of complex structures (continued)

Recall: (X,J) complex manifold

- J' near J can be described by $s \in \Omega^{0,1}(X, TX^{1,0})$
  namely $\Omega^{1,0}_{J'} = \mathrm{graph}(-s)$, $s: \Omega^{1,0}_J \to \Omega^{0,1}_J$
  $T^{0,1}_{J'} = \mathrm{graph}(s)$, $s: T^{0,1}_J \to T^{1,0}_J$
- Integrability of $J' \iff \bar\partial s + \frac{1}{2}[s,s] = 0$
- Action of $\phi \in \mathrm{Diff}(X)$ (near id): $\phi^* J$ is described by $s = -(\partial\phi)^{-1}\bar\partial\phi$
- Linearization (first-order deformations): $\bar\partial s_1 = 0$ mod $\mathrm{Im}\,\bar\partial$
  ie. $$\frac{\ker(\bar\partial : \Omega^{0,1}(X, TX^{1,0}) \to \Omega^{0,2})}{\mathrm{Im}(\bar\partial : C^\infty(X, TX^{1,0}) \to \Omega^{0,1})} = H^1_{\bar\partial}(X, TX^{1,0})$$

- Another way to think about this: (X,J) complex mfld $= \left(\bigsqcup U_i\right)/\phi_{ij}$
  U_i complex charts, $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ biholomorphisms, $\phi_{ji} = \phi_{ij}^{-1}$, $\phi_{ij}\phi_{jk} = \phi_{ik}$
  Then, deforming $(X,J) \iff$ deform gluing maps ϕ_{ij} among holom. maps
  to 1^{st} order, this is given by holom. vector fields v_{ij} on $U_i \cap U_j$
  & should satisfy $v_{ji} = -v_{ij}$, $v_{ij} + v_{jk} = v_{ik}$ on $U_i \cap U_j \cap U_k$
  $\Rightarrow$ Čech 1-cocycle with values in sheaf of holom. tangent vector fields

  Mod out by: $\psi_i : U_i \xrightarrow{\sim} U_i$ diffeo, change $\phi_{ij} \rightsquigarrow \psi_j \phi_{ij} \psi_i^{-1}$
  to 1^{st} order, v_i holom. vector fields on U_i, affect gluings by
  $v_{ij} = v_i - v_j$ ie. Čech coboundary

  $\rightsquigarrow$ get again $H^1(X, TX)$

- Obstruction: given a first-order deform^{n} s_1, can we find an actual deform^{n} $s(t) = s_1 t + O(t^2)$ (or a formal deform^{n} $\sum_{n\geq 1} s_n t^n$)?

Working order by order to solve $\bar\partial s(t) + \frac{1}{2}[s(t), s(t)] = 0$:

$$\bar\partial s_1 = 0$$
$$\bar\partial s_2 + \tfrac{1}{2}[s_1, s_1] = 0$$
$$\bar\partial s_3 + [s_1, s_2] = 0$$
$$\cdots\cdots$$

$\Rightarrow$ need: $[s_1, s_1] \in \operatorname{Im} \bar\partial \subseteq \Omega^{0,2}(X, TX^{1,0})$?

know: $[s_1, s_1] \in \ker \bar\partial$ (since $\bar\partial s_1 = 0$).

$\rightsquigarrow$ primary obstruction: class of $[s_1, s_1]$ in $\underline{H^2(X, TX^{1,0})}$.

If vanishes then $\exists\, s_2$ s.t. $\bar\partial s_2 + \frac{1}{2}[s_1, s_1] = 0$.

Next obstruction: class of $[s_1, s_2]$ in $H^2(X, TX^{1,0})$

If vanishes then $\exists\, s_3 \ldots$ and so on.

- If it happens that $H^2(X, TX) = 0$ then deformations are unobstructed ie. given s_1 lifts to all orders! ($\rightarrow \exists$ actual deformation).
  For Calabi-Yaus, in general $H^2(X,TX) \neq 0$, but remarkably:

<u>Thm</u> (Bogomolov-Tian-Todorov)

X compact Calabi-Yau with $H^0(X, TX) = 0 \Rightarrow$ deformations of X are unobstructed, ie., if $\operatorname{Aut}(X,J) = 1$, $\mathcal{M}_{cx}(X)$ is locally smooth w/ tangent space $\simeq H^1(X,TX)$.

for CY mfolds, $TX \simeq \Omega_X^{n-1}$ so
$v \mapsto \iota_v \Omega$

$$H^0(X,TX) = H^{n-1,0} \overset{\text{see below}}{\simeq} H^{0,1} \leftarrow \text{assume } 0$$
$$H^1(X,TX) \simeq H^{n-1,1} \quad \leftarrow \text{deform}^{\text{n}}$$
$$H^2(X,TX) \simeq H^{n-1,2} \quad \leftarrow \text{obstructions})$$

★ <u>Recall</u>: <u>Hodge theory</u> on compact Kähler manifolds :

- Kähler form := symplectic form compatible with cx. structure
  ie. ω real positive closed $(1,1)$-form; $g = \omega(\cdot, J\cdot)$ Herm metric
  (ie. $\omega = i \sum_{jk} \omega_{jk}\, dz_j \wedge d\bar z_k$, (ω_{jk}) positive definite Hermitian matrix)
  + require $d\omega = 0$; the corresponding metric is $g = \sum_{jk} \omega_{jk}\, dz_j\, d\bar z_k$

<u>Ex</u>: (<u>pointwise</u> always $\simeq$ this) $\mathbb{C}^n$, $\omega_0 = \sum dx_j \wedge dy_j = \frac{i}{2} \sum dz_j \wedge d\bar z_j$, $g_0 =$ Eucl.

③

- Kähler metric ⇝ $*$ operator : satisfies $\alpha \wedge * \bar{\beta} = g(\alpha,\beta)\, \mathrm{vol}_g$

  $*$: on real forms, in orthonormal basis, $\prod_{i\in I} dx_i \longmapsto \pm \prod_{i \notin I} dx_i$

  $\mathbb{C}$-linear extension maps $\Omega^{p,q} \to \Omega^{n-q,\,n-p}$

  <u>Ex:</u> in $(\mathbb{C}, g_0)$, $*(1) = dx\wedge dy = \frac{i}{2}\, dz \wedge d\bar{z}$ & vice versa

  $$*(dz) = *(dx + i dy) = dy - i dx = -i dz$$
  $$*(d\bar{z}) = i d\bar{z}.$$

- ⇝ adjoints $d^* = -*d*$, $\bar{\partial}^* = -*\partial*$ $\quad (\bar{\partial}^* : \Omega^{p,q} \to \Omega^{p,q-1}$ L^2-adjoint to $\bar{\partial})$

  Laplacians $\Delta = dd^* + d^*d$, $\quad \bar{\square} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$: $\underline{\Delta = 2\bar{\square}}$

  $\int |\Delta\alpha|^2 \mathrm{vol}_g = \int |d\alpha|^2 + |d^*\alpha|^2 \Rightarrow \operatorname{Ker}\Delta = \ker d \cap \ker d^*$
  $= \ker d \cap (\operatorname{Im} d)^\perp$

  Similarly with $\bar{\square}$

- Every cohomology class has a unique harmonic representative

  — " — $\bar{\partial}$-cohomology ——— " ——— $\bar{\square}$-harmonic — " —

  (<u>Hodge decomp. theorem</u>)

- so $H^k_{dR}(X,\mathbb{C}) \cong \operatorname{Ker}(\Delta : \Omega^k(X,\mathbb{C}) \circlearrowleft)$

  $$= \ker(\bar{\square} : \Omega^k(X,\mathbb{C}) \circlearrowleft)$$
  $$= \bigoplus_{p+q=k} \ker(\bar{\square} : \Omega^{p,q} \circlearrowleft) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X)$$

- $*$ gives $H^{p,q} \simeq H^{n-q,n-p}$; complex conj: $H^{p,q} \simeq \overline{H^{q,p}}$

  so Hodge diamond
  $$\begin{array}{ccccc} & & h^{n,n} & & \\ & & \vdots & & \\ h^{n,0} & & & & h^{0,n} \\ & h^{1,0} & h^{1,1} & & \\ & & h^{0,0} & & \end{array}$$
  is symmetric.

- For a CY n-fold, $H^{p,0} \underset{\text{Hodge}*}{\simeq} H^{n,n-p} = H^{n-p}(X,\Omega^n_X) \underset{CY}{\simeq} H^{n-p}(X,\mathcal{O}_X) = H^{0,n-p}$

  so $h^{p,0} = h^{n-p,0}$.

For a CY 3-fold
Under assumption $h^{1,0} = 0$, get

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

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* For Calabi-Yaus, we'll reinterpret Kodaira-Spencer map in terms of $[\Omega] \in H^{n,0} \subset H^n(X,\mathbb{C})$. For this we'll need:

<u>Thm.</u> (Griffiths transversality)

$$\alpha_t \in \Omega^{p,q}(X,J_t) \Rightarrow \frac{\partial}{\partial t}\alpha_t \in \Omega^{p,q} + \Omega^{p+1,q-1} + \Omega^{p-1,q+1}$$

<u>Pf.</u> (X,J_t) locally given by $s(t) \in \Omega^{0,1}(X, TX^{1,0})$, $s(0)=0$

In local coords, $T^*X^{1,0}_{J_t} = \mathrm{span}\{dz_i^{(t)} := dz_i - \sum_j s_{ij}(t)\, d\bar z_j\}$ (seen above)

$$\alpha_t = \sum_{|I|=p,|J|=q} \alpha_{IJ}(t)\, dz_{i_1}^{(t)} \wedge \dots \wedge dz_{i_p}^{(t)} \wedge d\bar z_{j_1}^{(t)} \wedge \dots \wedge d\bar z_{j_q}^{(t)}$$

Take $\frac{\partial}{\partial t}\big|_{t=0}$ & apply product rule: since $s_{ij}(0)=0$, only terms not $\in \Omega^{p,q}$

are $\alpha_{IJ}(0)\, dz_{i_1} \wedge \dots \wedge \left(\sum_j \frac{\partial s_{i_k j}}{\partial t} d\bar z_j\right) \wedge \dots \wedge dz_{i_p} \wedge d\bar z_{j_1} \wedge \dots \wedge d\bar z_{j_q} \in \Omega^{p-1,q+1}$

and similarly (differentiating $d\bar z_{j_k}$) terms in $\Omega^{p+1,q-1}$ ▲

• <u>Another interpretation</u> of Kodaira-Spencer map for Calabi-Yaus:

$$\begin{array}{ccc} \mathcal{X} & \supset & X \\ \downarrow & & \downarrow \\ S & \ni & 0 \end{array}$$

family of deformations of $(X,J) \rightsquigarrow (X,J_t)_{t\in S}$

$c_1(K_X)=0$ (deform. inv't) and $H^{0,1}=0$ assumed

$\Rightarrow K_{X_t} \simeq \mathcal{O}_{X_t}$ holomorphically even after deformation (so all (X,J_t) are Calabi-Yau)

Then $\exists [\Omega_t] \in H^{n,0}_{J_t}(X) \subset H^n(X,\mathbb{C})$. Q^n: how does it depend on t?

holom. vol. form

given $\frac{\partial}{\partial t} \in T_0 S$, $\frac{\partial}{\partial t}\Omega_t \in \Omega^{n,0} \oplus \Omega^{n-1,1}$ by Griffiths transversality

Now: $\frac{\partial \Omega_t}{\partial t}$ is d-closed (since Ω_t d-closed)

$\Rightarrow \left(\frac{\partial \Omega_t}{\partial t}\right)^{(n-1,1)}$ is $\bar\partial$-closed $\Rightarrow \exists \left[\frac{\partial \Omega_t}{\partial t}^{(n-1,1)}\right] \in H^{n-1,1}(X)$

* As seen above, $H^{n-1,1}(X) = H^1(X,\Omega_X^{n-1}) \simeq H^1(X,TX)$

the image of $\left[\frac{\partial \Omega_t}{\partial t}\right]$ in $H^1(X,TX)$ is indep't of choices and $\equiv$ Kodaira-Spencer map.