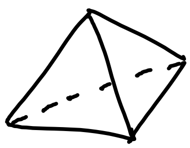
① • <u>general approach to Slag fibr^ns</u>: <u>Toric degenerations</u> (= special type of LCSL)
(Haase–Zharkov, WD Ruan, Gross–Siebert, ...)

<u>Idea</u>: degenerate X to union of toric varieties, build degenerate fibration there, try to smooth?

<u>Sketch in K3 case</u>: $X_\lambda = \{P_\lambda := x_0x_1x_2x_3 + \lambda P_4(x_0:\dots:x_3) = 0\} \subset \mathbb{CP}^3$

$$\omega_\lambda = \omega_{\mathbb{CP}^3}|X_\lambda, \quad \Omega_\lambda = \mathrm{Res}_{X_\lambda}\left(\frac{dx_1\,dx_2\,dx_3}{P_\lambda}\right)$$



As $\lambda \to 0$, degenerates to $X_0 = \bigcup 4\ \mathbb{CP}^2$'s.

on each component ω_0 = standard, $\Omega_0 = \prod \frac{dx_i}{x_i}$

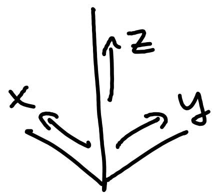
Product tori $\{|x_i| = \text{const}\}$ are special Lagrangian ($T^2 \subset \mathbb{CP}^2$), but degenerate to S^1 at edges, pt at vertices.

Smooth this ($\lambda \neq 0$ small) ??

Model in dim 1: $\{xy = 0\} \subset \mathbb{C}^2$ smooth to $\{xy = \lambda\}$,

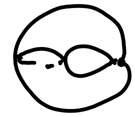
$\Omega = \frac{dx}{x} = -\frac{dy}{y}$ $\rightsquigarrow$ circles $|x| = \text{cst}$, $|y| = \text{cst}$ are Slag.

In dim. 1 more, model along edge = S^1 times this:



$|z| = \text{cst} \rightsquigarrow S^1_z$ times this model

except ... perturb $xy = 0 \to xy + \lambda P_4(z) = 0$
four roots

those become 

$\Rightarrow$ 4 sing^s of T^2-fibration on each edge of 

get S^2 with affine structure on $S^2 - \{24 \text{ pts}\}$

★ The same procedure holds in greater generality, gives affine structures & way of building a candidate mirror (Gross–Siebert)

However not clear if the affine mfld built this way is the base of a Slag fibration (probably <u>not</u> [Joyce])

② Landau-Ginzburg models & non-CY manifolds:

Motivating example: $\mathbb{CP}^1$ is mirror to $\left(\mathbb{C}^*, \ W = z + \frac{1}{z}\right)$??

Landau-Ginzburg model = noncompact Kähler mfld + holomorphic function W ("superpotential")

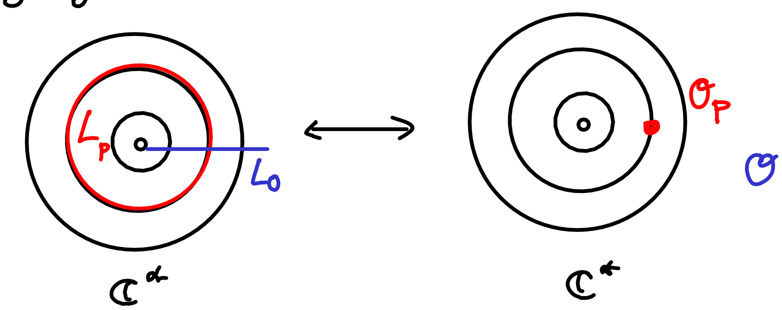
W measures "obstruction to being CY" and affects geometric interpretation of mirror symmetry. Slogan: geometry of $X \longleftrightarrow$ geometry of $\mathrm{crit}(W) \subset \check{X}$.

In our example:

Start with $\mathbb{C}^*$, (ω = any), $\Omega = \frac{dz}{z}$: (open Calabi-Yau)

SLag fibration by circles $S^1(r) = \{|z| = r\}$, base $\cong \mathbb{R}$

Dualizing gives back $\mathbb{C}^*$



mirror symmetry à la SYZ works well

(e.g. $HF(L_p, L_p) \simeq H^*(S^1, \mathbb{C}) \simeq \mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p)$)

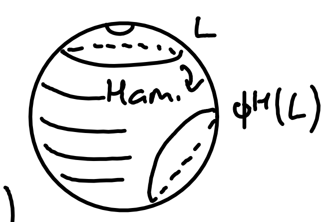
however need to incorporate noncompact Lagrangians

[Seidel's "wrapped Fukaya category": perturb by rotation at ∞:

$HW(L_0, L_0) \simeq \mathbb{C}[t^{\pm 1}] \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{O})$ (holom. functions over $\mathbb{C}^*$)].

Now look at $\mathbb{CP}^1 = \mathbb{C}^* \cup \{0, \infty\}$, ω = std., $\Omega = \frac{dz}{z}$ (with poles at 0 & ∞)

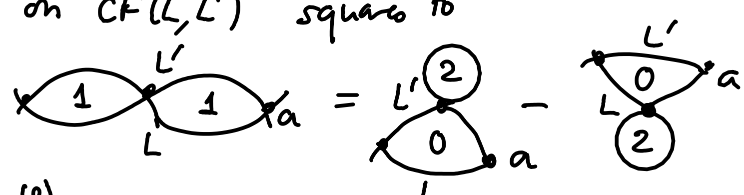
then can still consider family of SLag circles, but typically $HF(L, L) = 0 \rightsquigarrow$ zero object in $D^b\mathrm{Fuk}$



Also: Floer homology is <u>obstructed</u>! (circles bound discs !!)

Recall: when L, L' bound discs, ∂ on $CF(L, L')$ squares to

$\partial^2(a) = m_0' \cdot a - a \cdot m_0$:



$$m_0 = \sum_{\beta \in \pi_2(X, L)} ev_* \left[\overline{\mathcal{M}}_1(X, L; J, \beta)\right] T^{\omega(\beta)} \mathrm{hol}_\nabla(\partial\beta) \in CF(L, L)$$

holom. discs with one ∂ marked point.

③ These features of Floer homology are encoded in the superpotential.

Namely: $X = \mathbb{CP}^1$ Kähler mfld, $D = \{0,\infty\}$ anticanonical divisor $\left(s_D \in H^0(K_X^{-1})\right)$

$\Omega = s_D^{-1} \in H^0(X \setminus D, K_X)$ here $\Omega = \frac{dz}{z}$ on $\mathbb{C}^*$

$\rightsquigarrow M = \{(L,\nabla) \,/\, L$ SLag torus in $X \setminus D$, ∇ flat $U(1)$-conn.$\}$

SYZ mirror to almost-CY manifold $X \setminus D$.

$L \subset X \setminus D$ SLag, $\beta \in \pi_2(X,L) \rightsquigarrow$ Maslov index $\mu(\beta) = 2 \cdot (\beta \cap D)$

(Note: s_D gives a trivialization of $\det(TM)$ away from D .....).

Expected dimension of $\overline{\mathcal{M}}(X,L,J,\beta) = n - 3 + \mu(\beta)$

In our case, positivity of intersection $\Rightarrow \mu(\beta) \geq 0$ for holom. discs

<u>Assume</u>: • $\nexists$ nonconstant $\mu = 0$ holom. discs in (X,L) – ie. all discs hit D.

OK for $\mathbb{CP}^1$ (no discs in $(\mathbb{C}^*, S^1(r))$ by max. principle)

• $\mu = 2$ discs (hitting D once) are regular (also ok for $\mathbb{CP}^1$)

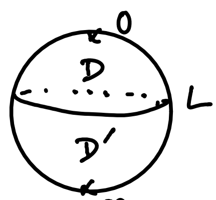
(These assumptions are ok for toric Fano mflds, e.g. $\mathbb{CP}^n$'s & products)

Then $\mu = 2$ moduli spaces are compact (no bubbling of discs), dim. $n-1$

Can define $n_\beta = \deg\left(ev_{0*}[\overline{\mathcal{M}}_1(\beta)]\right)$ = "# holom. discs in class β whose boundary contains a generic pt $\in L$" $\in \mathbb{Z}$

<u>Define</u> $\Bigg\|$ $W(L,\nabla) := \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta) = 2}} n_\beta \, z_\beta(L,\nabla)$ where $z_\beta = e^{-2\pi \int_\beta \omega} \mathrm{hol}_{\partial\beta}(\nabla)$

In our example:



Two $\mu = 2$ discs D and D'

D contributes z

D' — " — z'

Relation: $[D] + [D'] = [\mathbb{CP}^1] \Rightarrow zz' = e^{-2\pi \int_{\mathbb{CP}^1} \omega} =: e^{-\Lambda}$

Hence $W = z + z' = z + \dfrac{e^{-\Lambda}}{z}$.

<u>Homological mirror symmetry</u>: (M. Kontsevich '98):

$$\begin{cases} D^\pi \mathcal{F}uk(\mathbb{CP}^1) \simeq H^0 MF(W) & \text{matrix factorizations} \\ D^b Coh(\mathbb{CP}^1) \simeq D^b \mathcal{F}uk(\mathbb{C}^*, W). & \text{"Fukaya-Seidel" category} \end{cases}$$

The first one explains our construction of the mirror.

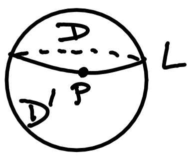
(4) Fukaya category: actually a collection indexed by "charge" $\lambda \in \mathbb{C}$.

$\mathcal{F}uk(\mathbb{CP}^1, \lambda) = \{$ weakly unobstructed Lagrangians with $m_0 = \lambda . [L] \}$

This is an honest A_∞-cat. (m_0's cancel, Floer differential $\partial^2 = 0$) whereas from λ to λ' we'd have $\partial^2 = \lambda' - \lambda$.

Ex: $(L = \text{circle}, \nabla)$ is weakly unobstructed, $m_0 = W(L,\nabla).[L]$

However: $HF(L,L) = 0$ unless $L =$ equator & $hol(\nabla) = \pm id$.



$$\partial([p]) = z . ev_{0*}([\mathcal{M}_2(L,[D])] \cap ev_1^{-1}(p)) + z' . (\text{same with } D')$$
$$= z . [L] - z' . [L] .$$

Hence unit $[L] \in \operatorname{im} \partial$ unless $z = \frac{e^{-\Lambda}}{z}$ ie. $z = \pm e^{-\Lambda/2}$
ie. (equator, $\pm$).

For (equator, $\pm$), contrib^{ns} of pairs of symmetric discs cancel exactly and $HF^*(L,L) \simeq H^*(S^1; \mathbb{C})$ as a $\mathbb{Z}/2$-graded vector space

However product structure is deformed: $m_2([p],[p]) = \pm e^{-\Lambda/2} [L]$

ie. multiplicatively $HF^*(L,L) \simeq \mathbb{C}[t]/t^2 = \pm e^{-\Lambda/2}$.