1. **Recall:** want to check HMS for the elliptic curve. [Polishchuk–Zaslow]

- $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\omega = \lambda\, dx \wedge dy$ $\left(\int_{T^2}\omega = \lambda\right)$
  $X^\vee = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, $\tau = i\lambda$.

- straight line Lagrangians in X of slope q/p (+ flat $U(1)$ conn.) $\longleftrightarrow$ rank p, degree $-q$ holom. bundles over $X^\vee$

- A degree 1 line bundle $\mathcal{L} \to X^\vee$ is given by:
  $$\mathcal{L} \simeq \mathbb{C}\times\mathbb{C} \Big/ \begin{array}{l}(z,v)\sim(z+1,v)\\ (z,v)\sim(z+\tau, e^{-\pi i \tau} e^{-2\pi i z} v)\end{array}$$
  $\vartheta(\tau,z) = \vartheta[0,0](\tau,z)$ holom. section of $\mathcal{L}$

  $\vartheta[\frac{k}{n},0](n\tau, nz)$ $k=0,\dots,n-1$ basis of holom. sections of $\mathcal{L}^{\otimes n}$

where $\vartheta[c',c''](\tau,z) := \sum_{m\in\mathbb{Z}} \exp 2\pi i\left(\tau\frac{(m+c')^2}{2} + (m+c')(z+c'')\right)$

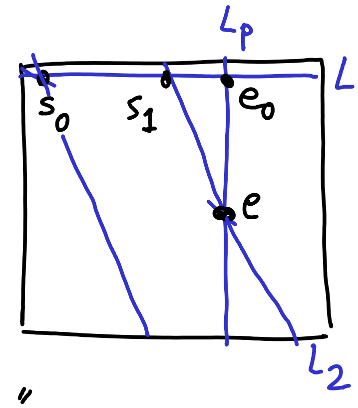
$$\left(\begin{array}{ll}\text{recall} & \theta[c',c''](\tau,z+1) = e^{2\pi i c'}\theta[c',c''](\tau,z)\\ & \theta[c',c''](\tau,z+\tau) = e^{-\pi i\tau}e^{-2\pi i(z+c'')}\theta[c',c''](\tau,z)\end{array}\right)$$

<u>A first check:</u> $L_0 = \{(x,0)\}$, $\nabla_0 = d$
"mirror to $\mathcal{O}$"

$L_n = \{(x,-nx)\}$, $\nabla_n = d$
"mirror to $\mathcal{L}^{\otimes n}$"

$L_p = \{(a,y)\}$, $\nabla_p = d + 2\pi i b\, dy$
"mirror to $\mathcal{O}_z$, $z = b + a\tau$"

e.g.:



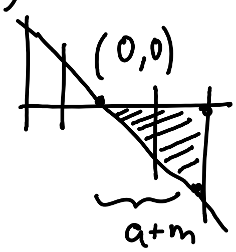
$$\left.\begin{array}{lll}\text{grade } (L_0,\nabla_0) & \text{so} & \arg(dz_{|TL_0}) = 0\\ (L_n,\nabla_n) & & \in(-\frac{\pi}{2},0)\\ (L_p,\nabla_p) & & = -\pi/2\end{array}\right\}$$
Then $s_k = (\frac{k}{n},0) \in CF^0(L_0,L_n)$
$e = (a,-na) \in CF^0(L_n,L_p)$
$e_0 = (a,0) \in CF^0(L_0,L_p)$
are all in <u>degree 0</u>.

<u>Compute:</u> $m_2(e_n, s_0) = \underline{\ ?\ }\, e_0$?
$\uparrow$ need to count holomorphic discs in T^2...

②

Observe:

- discs lift to universal cover $\mathbb{R}^2 = \mathbb{C}$
- Maslov index calculation $\Rightarrow$ rigid holom. discs are immersed (as maps $D^2 \to \mathbb{C}$, derivative has no zeroes)
- get an ∞ sequence of holom. triangles T_m, $m \in \mathbb{Z}$ in univ. cover vertices are at $(0,0)$, $(a+m, -n(a+m))$, $(a+m, 0)$.



$\Rightarrow$ area $\int_{T_m} \omega = \lambda n (a+m)^2/2$

- holonomy / boundary is $\exp\left(2\pi i \int_{-n(a+m)}^{0} b\, dy\right) = \exp\left(2\pi i n (a+m) b\right)$
- the immersed triangles T_m are all regular (calc. $\bar\partial$-operator...)
- sign calc^{n}: orient^{n} is $+1$ for all T_m (if trivial spin structures)

$$\Rightarrow m_2(e, s_0) = \underbrace{\left(\sum_{m \in \mathbb{Z}} T^{\frac{\lambda n (a+m)^2}{2}} e^{2\pi i n (a+m) b}\right)}_{\text{set } T = e^{-2\pi},\ \text{i.e. } T^\lambda = e^{2\pi i \tau}} e_0$$

$$\Rightarrow \text{coeff}^t = \sum_{m \in \mathbb{Z}} \exp 2\pi i \left(\frac{n\tau m^2}{2} + n(\tau a + b) m + \left(\frac{n\tau a^2}{2} + nab\right)\right)$$

$$= \underbrace{e^{\pi i n \tau a^2} e^{2\pi i n a b}}_{} \theta(n\tau, n(\tau a + b))$$

change of triv^n (holomorphic vs. unitary) at $z = \tau a + b$

i.e.: this is $\mathcal{O} \xrightarrow{s_0} \mathcal{L}^n \xrightarrow{ev_z} \mathcal{O}_z$

ev_z : evaluation at z in <u>suitable trivialization</u> - not the one we thought!

To check if this is valid, try same with s_k (intersection at $(\frac{k}{n}, 0)$):

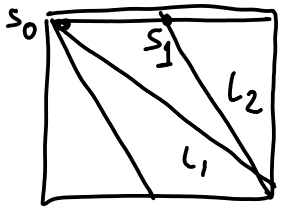
coeff^t of e_0 in $m_2(e, s_k)$ is similarly

$$\sum_{m \in \mathbb{Z}} \exp 2\pi i \left(\frac{n\tau}{2}\left(a+m-\frac{k}{n}\right)^2 + n\left(a+m-\frac{k}{n}\right) b\right)$$

$$= \sum_{m \in \mathbb{Z}} \exp 2\pi i \left(\frac{n\tau}{2}\left(m-\frac{k}{n}\right)^2 + n(\tau a + b)\left(m - \frac{k}{n}\right) + \frac{n\tau a^2}{2} + nab\right)$$

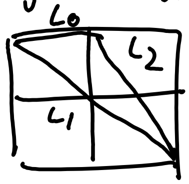
$$= e^{\pi i n \tau a^2} e^{2\pi i n a b}\, \theta\left[-\tfrac{k}{n}, 0\right](n\tau, n(\tau a + b))$$ i.e. ratios match $\frac{s_k(z)}{s_0(z)}$ ✓

③ · Similarly, look at <u>multiplication</u> of sections: simplest example:



$$m_2(\underset{\hom(L_1,L_2)}{s_0}, \underset{\in \hom(L_0,L_1)}{s_0}) = \underset{\hom(L_0,L_2)}{c_0 s_0 + c_1 s_1} \; ?$$

c_0 counts triangles with all vertices at s_0, there's a constant one then



area $= \lambda$, and others ...

$$\rightsquigarrow c_0 = \sum_{n\in\mathbb{Z}} T^{n^2\lambda} = \sum_{n\in\mathbb{Z}} e^{2\pi i \tau n^2}$$

similarly $c_1 = \sum_{n\in\mathbb{Z}} e^{2\pi i\tau(n+\frac{1}{2})^2}$

corresponds to: $\theta(\tau,z)\,\theta(\tau,z) = \underbrace{\theta[0,0](2\tau,0)}_{c_0}\,\underbrace{\theta[0,0](2\tau,2z)}_{s_0} + \underbrace{\theta[\tfrac{1}{2},0](2\tau,0)}_{c_1}\,\underbrace{\theta[\tfrac{1}{2},0](2\tau,2z)}_{s_1}$

Can do more systematic calculations for more general line bundles and also higher rank bundles $\rightsquigarrow$ build a functor between homology categories & check m_2 is preserved. [Polishchuk-Zaslow]

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* To actually prove HMS, need to understand (& match) leftover part of A_∞-structure on derived category: <u>Massey products</u>.

Look at a special case: in a tri. cat. $\mathcal{D}$, consider objects & morphisms $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} X_4$ where $g\cdot f=0$, $h\cdot g=0$,
& assume also $\hom(X_1, X_3[-1]) = \hom(X_2, X_4[-1]) = 0$

We still have an element $m_3(h,g,f) \in \hom(X_1, X_4[-1])$:

let K be s.t.
$$\begin{tikzcd} K \arrow[r] & X_2 \arrow[dl, "g"] \\ X_3 \arrow[u, "{[1]}"] & \end{tikzcd}$$
distinguished (ie. $K[1] = \mathrm{Cone}(g)$)

then $g\cdot f = 0 \Rightarrow f$ factors through $X_1 \xrightarrow{\bar{f}} K \to X_2$
$h\cdot g = 0 \Rightarrow h$ factors through $X_3 \to K[1] \xrightarrow{\bar{h}} X_4$

[argument: $\hom(X_1,K) \to \hom(X_1,X_2) \xrightarrow{g} \hom(X_1,X_3)$ exact $\Rightarrow f$ factors
also $\hom(X_1,X_3[-1]) = 0 \Rightarrow$ factors uniquely].

Now $m_3(h,g,f) := m_2(\bar{h}[-1], \bar{f})$: $\quad X_1 \xrightarrow[\bar{f}]{} K \xrightarrow[\bar{h}[-1]]{} X_4[-1]$

* Why is that related to m_3 from A_∞ structure?

Lift f,g,h to "chain level" A_∞-tri. cat. of (twisted) complexes, then can take $K = \{ X_2 \xrightarrow{g} X_3[-1] \}$ and now $\bar{f}, \bar{h}[-1]$ are

$$\begin{tikzcd} X_1 \arrow[d, "f"'] & & \\ X_2 \arrow[r, "g"] & X_3[-1] \arrow[d, "h[-1]"] & \\ & X_4[-1] & \end{tikzcd}$$

$m_2^{Tw}(\bar{h}[-1], \bar{f}) = m_3(h,g,f)$ by def^{n} of m_2^{Tw} (insert δ's everywhere).

* Look at: $\mathcal{L}$ nontrivial degree 0 line bundle $\mathcal{L} \to X^\vee$ $\qquad p, q \in X^\vee$ distinct generic

$$\mathcal{O} \xrightarrow[f]{\text{1-dim'l}} \mathcal{O}_p \xrightarrow[g]{\text{1-dim'l}} \mathcal{L}[1] \xrightarrow[h]{\text{1-dim'l}} \mathcal{O}_q[1]$$

$\hom(\mathcal{O}_p, \mathcal{L}[1]) = \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{L}) \underset{\text{Serre}}{\simeq} \mathrm{Hom}(\mathcal{L}, \mathcal{O}_p)^\vee \simeq$ fiber of $\mathcal{L}$ at p

$\hom(\mathcal{O}, \mathcal{L}[1]) = \mathrm{Ext}^1(\mathcal{O}, \mathcal{L}) = H^1(\mathcal{L}) = 0$ by Riemann-Roch

$\hom(\mathcal{O}_p, \mathcal{O}_q[1]) = 0$

Massey product of generators: $K \simeq \underbrace{\mathcal{L} \otimes \mathcal{O}(p)}_{\text{another deg 1 line bundle}}$ $\leftarrow$ deg 1 line bundle w/ section vanishing at p

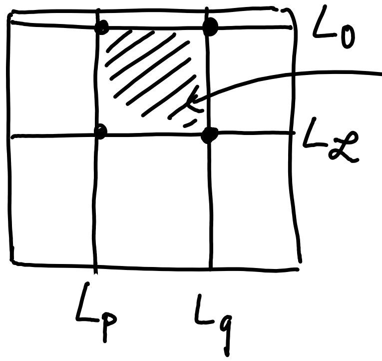
$\left(0 \to \mathcal{L} \xrightarrow[s_p]{} \mathcal{L} \otimes \mathcal{O}(p) \to \mathcal{O}_p \to 0 \right.$ + rotate exact triangle$\left.\right)$
has the extension class g $\quad \Rightarrow$

$$\begin{tikzcd} K \arrow[r] & \mathcal{O}_p \arrow[dl, "g"] \\ \mathcal{L}[1] \arrow[u, "{[1]}"] & \end{tikzcd}$$

Hence: $\bar{f}$ = nontrivial section of deg 1 bundle $K \simeq \mathcal{L} \otimes \mathcal{O}(p)$
$\bar{h}[-1]$ = nontrivial hom from K to $\mathcal{O}_q$ (or rather $K[1] \to \mathcal{O}_q[1]$)
(as long as $\mathcal{L} \otimes \mathcal{O}(p) \not\simeq \mathcal{O}(q)$).

$\Rightarrow$ This Massey product is nontrivial, and can be computed and compared with the Fukaya cat. m_3:

⑤



$m_3: L_0 \to L_p \to L_\mathcal{L}[1] \to L_q[1]$.

This rectangle is one in a $\mathbb{Z}^2$-family of rectangles that contribute to m_3

[see Polishchuk].

With more work one can prove HMS for T^2 in this way ...